ODE 2.

Recall: $f: C^1$ near (x_0, y_0)

$$\Downarrow$$

$$(*) \begin{cases} y'(x) = f(x, y(x)) \\ y(x_0) = y_0 \end{cases}$$

has a unique solution $y(x)$ near x_0.

Existence and uniqueness theorem for ordinary differential equation.

$\Uparrow$ idea: Fixed point iteration. (Banach fixed point theorem)

wiki page

$$(*) \iff y(x) - y_0 = \int_{x_0}^{x} f(t, y(t))\, dt \quad (**)$$

$$\left(\because \begin{cases} y'(t) = f(t, y(t)) \\ y(x_0) = y_0 \end{cases} \iff \begin{cases} \int_{x_0}^{x} y'(t) = \int_{x_0}^{x} f(t, y(t)) \\ y(x_0) = y_0 \end{cases} \right.$$

$$\left. \iff y(x) - y_0 = \int_{x_0}^{x} f(t, y(t)) \right)$$

Aim: Use $(**)$ to find numerical approximation of $y(x)$.

Step 1: Let $x = x_0 + h$

$$\Downarrow (**)$$

$$y(x_0+h) - y_0 = \int_{x_0}^{x_0+h} f(t, y(t))\, dt$$

Step 2: For small h, $t \in (x_0, x_0+h)$ close to x_0

$$\Rightarrow f(t, y(t)) \underset{\text{close}}{\sim} f(x_0, y(x_0)) = f(x_0, y_0)$$

Step 3: Put y_1 $(\sim y(x_0+h))$

$$y_1 = y_0 + h \cdot f(x_0, y_0)$$

Now let us continue:

Step 4: Put $x_1 = x_0 + h$. Consider $(*_1): \begin{cases} y' = f(x, y) \\ y(x_1) = y_1 \end{cases}$

$$\Downarrow$$

Euler method:

$$y_{k+1} = y_k + h\, f(x_k, y_k)$$

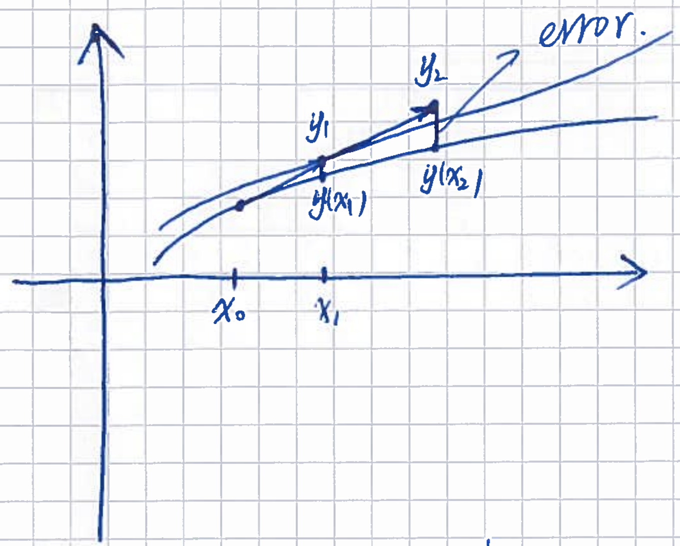
Start from (x_0, y_0)

$x_{k+1} := x_k + h.$

$y_{k+1} \approx y(x_{k+1})$

Euler approximation!

Picture:



Examples: (See the Jupyter notes)

$$\begin{cases} y'(x) = -2x\, y(x) \\ y(0) = 1 \end{cases} \qquad \begin{cases} y' = y(1-y) \\ y(0) = y_0 \end{cases}$$

System of ODEs

Example: Lotka-Volterra (Jupyter notes for the numericals)

$$\begin{cases} y_1' = \alpha y_1 - \beta y_1 y_2 \\ y_2' = \delta y_1 y_2 - \gamma y_2 \end{cases}$$

y_1: number of preys

y_2: —— predators.

Higher order ODEs $\longleftrightarrow$ System of ODEs

Example: Newtonian mechanics

$F = ma$

mass

$a = \ddot{x} = -g$

↳ acceleration of the motion.

gravitational ~~constant~~ acceleration

$\simeq 9.8$

$$\ddot{x} = -g \qquad \begin{array}{l} x(0) = x_0 \\ \dot{x}(0) = x_0' \end{array}$$

Let $y(t) := \dot{x}(t) \Rightarrow \dot{y} = \ddot{x}$

then we get

$$(N) \begin{cases} \dot{x} = y & x(0) = x_0 \\ \dot{y} = -g & y(0) = x_0' \end{cases}$$

One second order ODE $\Leftrightarrow$ Two first order ODEs.

$\Leftrightarrow$ Hamiltonian mechanics description.

Put $\begin{cases} p \text{ (momentum)} := m\dot{x} = my \\ H \text{ (Hamiltonian total energy)} \\ \quad := \dfrac{p^2}{2m} + mgx \end{cases}$

we get

$$(N) \Leftrightarrow (H) \begin{cases} \dot{x} = \dfrac{\partial H}{\partial p} \\ \dot{p} = -\dfrac{\partial H}{\partial x} \end{cases}$$

Further example (Van der Pol oscillator : see the notes)

Error analysis :

Solve ODE on $[x_0, x_{end}]$

$$x_{end} - x_0 := N \cdot h$$

$$x_{end} = x_N$$

Estimate $\underline{y(x_N) - y_N} := e_N$!

Numerical example: (Notes Jupyter)

$$\Downarrow$$

$$|y(x_N) - y_N| \le c \cdot h \quad \text{(Euler's method)}$$

Theorem: Assume $|f_y| \le L$, $|y''| \le 2D$

$$\Downarrow$$

$$|y(x_N) - y_N| \le D \cdot \frac{e^{L(x_N - x_0)} - 1}{L} \cdot h$$

(NOT nice because we don't know y, how to estimate $|y''|$ ...).

Def: Method is of order p $\triangleq$ $|y(x_N) - y_N| < c \cdot h^p$.

$$h = \frac{x_{end} - x_0}{N}.$$

Example (Order 2 Heun's method)

Start from (x_0, y_0)

$$\left.\begin{aligned} k_1 &:= f(x_n, y_n) \\ k_2 &:= f(x_n + h, y_n + h k_1) \\ y_{n+1} &:= y_n + \frac{h}{2}(k_1 + k_2) \end{aligned}\right\} \to \text{comes from trapezoidal rule } \int_0^1 f \sim \frac{1}{2}(f(0) + f(1))$$

$$x_{n+1} = x_n + h \quad \cdots$$

Example: (Order 4: classical Runge-Kutta method)

$k_1 = f(x_n, y_n)$

$k_2 = f(x_n + \frac{h}{2}, y_n + \frac{h}{2} k_1)$

$k_3 = f(x_n + \frac{h}{2}, y_n + \frac{h}{2} k_2)$

$k_4 = f(x_n + h, y_n + h k_3)$

$$y_{n+1} := y_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$\uparrow$ comes from simpson's rule $\int_0^1 f \sim \frac{1}{6}(f(0) + 4f(\frac{1}{2}) + f(1))$