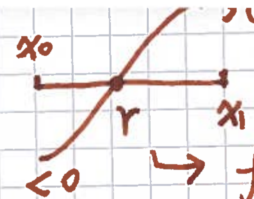
Recall ①

solve $f(x)=0$ $\begin{cases} \text{Bisection} \\ \underline{\text{fixed point}} \end{cases}$



$\rightarrow f(r)=0$

$\underline{x_{k+1} = g(x_k)}$

$\updownarrow$

: $f(x)=0 \iff x=g(x)$: fixed point equation

How to find optimal g ?

$g = x - \frac{f(x)}{f'(x)} \rightsquigarrow$ Newton method

$$\underline{x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}}$$

System of non-linear equations

$$\underline{\vec{f}(\vec{x})=0}$$

$$\updownarrow$$

$$\begin{cases} f_1(x_1, \cdots, x_n)=0 \\ \\ f_n(x_1, \cdots, x_n)=0 \end{cases}$$

We use newton method (special fixed point method)

$\downarrow$

$$\vec{x}_{k+1} = \vec{x}_k - \underline{J(x_k)}^{-1}\vec{f}(\vec{x}_k)$$

: $\underline{\text{Jacobian matrix}}$

$$\| \triangleright$$

$$\begin{pmatrix} \partial f_1/\partial x_1 & \cdots & \partial f_1/\partial x_n \\ \vdots & & \\ \partial f_n/\partial x_1 & \cdots & \partial f_n/\partial x_n \end{pmatrix}$$

Example : $\begin{cases} x_1^3 - x_2 + \frac{1}{4}=0 \\ x_1^2 + x_2^2 - 1 = 0 \end{cases}$

$\vec{x}_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\overrightarrow{f(x_0)} = \begin{pmatrix} \frac{1}{4} \\ 1 \end{pmatrix} \Leftarrow$ $\begin{cases} J = \begin{pmatrix} 3x_1^2 & -1 \\ 2x_1 & 2x_2 \end{pmatrix} \\ \Downarrow \\ J^{-1} = \dfrac{\begin{pmatrix} 2x_1 & 1 \\ -2x_1 & 3x_1^2 \end{pmatrix}}{6x_1^2x_2 + 2x_1} \end{cases}$

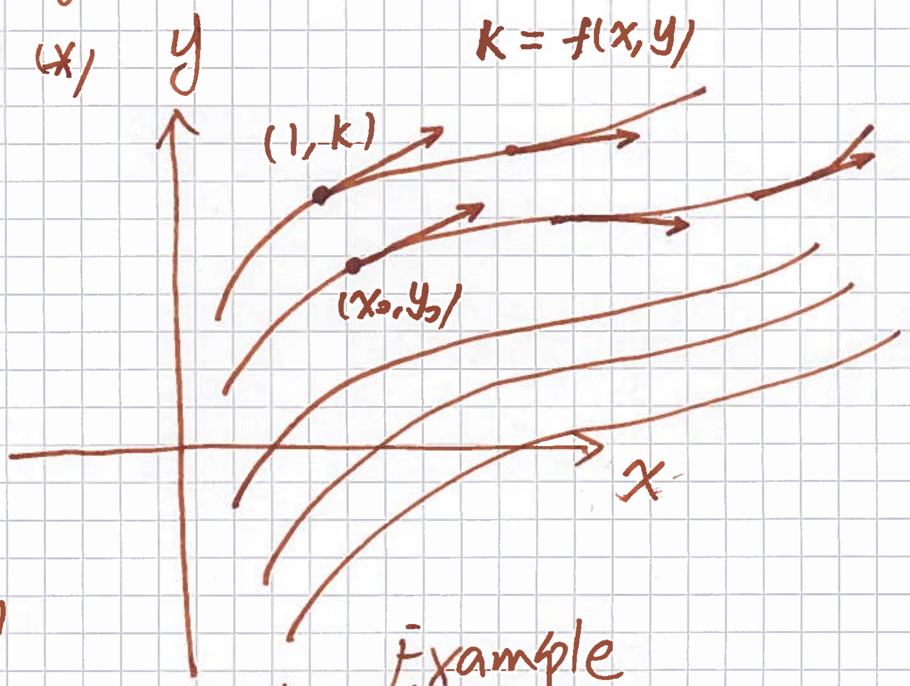
$$\vec{x}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \frac{\begin{pmatrix} 2 & 1 \\ -2 & 3 \end{pmatrix}}{8}\begin{pmatrix} \frac{1}{4} \\ 1 \end{pmatrix}$$

$= \cdots$

② ODE

Existence and uniqueness theorem for ODEs! (NOT Assumed)

$$\begin{cases} y'(x) = \underline{f(x, y(x))} & (*) \\ y(x_0) = y_0 \end{cases}$$



Question: explain the examples.

Idea: (Picard) $\int_{x_0}^{x} y'(\overset{t}{x})\,d\overset{t}{x}$

$$\| $$

$$y(x) - y(x_0)$$

$$\| $$

$$\underline{y(x) - y_0}$$

$$(*) \Updownarrow$$

$$y(x) - y_0 = \int_{x_0}^{x} f(x, y(x))\,dx \quad (**)$$

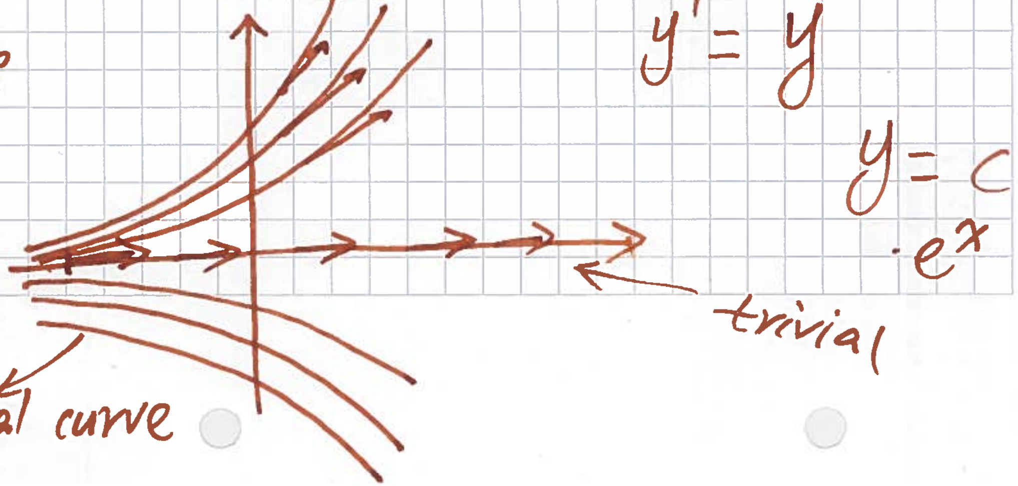
$$F(y)(x) = y_0 + \int_{x_0}^{x} f(x, y(x))\,dx$$

$$\uparrow$$

$(**)$: $y = F(y)$

Example

$y' = y$



Example : ③

$$\begin{cases} y' = 1 + y^2 \\ \underline{y(0) = 0} \end{cases} \quad \Bigg| \quad \underline{y = \tan x = \frac{\sin x}{\cos x}}$$

Picard iteration : $\underline{F(y) = 0 + \int_0^x 1 + y^2}$.

$$\varphi_0 \equiv y(0) = 0$$

$$\varphi_1 = F(\varphi_0) = \int_0^x 1 + 0^2 = \underline{x} \quad (\varphi_1(t) = t)$$

$$\varphi_2 = F(\varphi_1) = \int_0^x 1 + t^2 \, dt = x + \frac{x^3}{3}$$

$$\varphi_3 = F(\varphi_2) = \int_0^x 1 + \left(t + \frac{t^3}{3}\right)^2 dt$$

$$= \int_0^x 1 + t^2 + \frac{2}{3} t^4 + \frac{t^6}{9} \, dt$$

$$\vdots \qquad = x + \frac{x^3}{3} + \frac{2}{15} x^5 + \frac{x^7}{63}$$

φ_∞ : taylar expansion of $\underline{\tan x}$!

④ Picard iteration (fixed point iteration)

$\varphi_0 \equiv y_0$

$\varphi_1 := F(\varphi_0)$

$\vdots$

$\underline{\varphi_{k+1} := F(\varphi_k)}$

$$\varphi_{k+1}(x) = y_0 + \int_{x_0}^{x} f(t, \varphi_k(t))\, dt$$

$\Downarrow$ Estimate $\underline{\varphi_{k+1} - \varphi_k}$

$$\varphi_{k+1}(x) - \varphi_k(x) = \int_{x_0}^{x} \underline{f(t, \varphi_k(t)) - f(t, \varphi_{k-1}(t))}\, dt$$

Ⓛ $|f(t, \varphi_k) - f(t, \varphi_{k-1})| \le L\,|\varphi_k - \varphi_{k-1}|$

$\underline{f : C^1 \text{ w.r.t. } y}$ $\mid$ $L \sim \max |f_y|$

$\nearrow$ $\underline{\text{Lipschitz condition.}}$

Ⓛ $\Rightarrow$ $|\varphi_{k+1}(x) - \varphi_k(x)| \le L \underline{\int_{x_0}^{x} |\varphi_k(t) - \varphi_{k-1}(t)|\, dt}$

$\le \underline{L \cdot |x - x_0|} \cdot \max |\varphi_k - \varphi_{k-1}|$

$\downarrow$ small

When $\underline{|x - x_0| \cdot L < \frac{1}{2}}$ $\Leftarrow$

$\|\varphi_{k+1} - \varphi_k\|_\infty = \underset{\cancel{[t]}}{\max} |\varphi_{k+1} - \varphi_k|$

$\|\varphi_{k+1} - \varphi_k\|_\infty \overset{\uparrow L}{\le} \frac{1}{2} \|\varphi_k - \varphi_{k-1}\|$ (Estimate)

⑤ Estimate $\Rightarrow$

$$\| \varphi_{k+n} - \varphi_{\underline{k}} \| \leq \| \varphi_{k+n} - \varphi_{k+n-1} \| + \cdots + \underline{\| \varphi_{k+1} - \varphi_k \|}$$

$$\leq \| \varphi_{k+1} - \varphi_k \| \underline{\left(1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \cdots + \left(\frac{1}{2}\right)^{n-1} \right)}$$

$$\leq 2 \| \varphi_{k+1} - \varphi_k \|$$

$$\leq 2^{-k+1} \| \varphi_1 - \varphi_0 \| \to 0 \; (k \to \infty)$$

$\Downarrow$

$\{ \varphi_k(x) \}$ is a <u>cauchy sequence</u>, for all $\underline{|x - x_0|} < \frac{1}{2L}$.

$\Downarrow$

$\lim_{k \to \infty} \varphi_k(x) :\triangleq \underline{\varphi(x)}$ $\Big| \Rightarrow \exists \; \& \; 1$

<u>Summary</u>: For $|x - x_0| < \frac{1}{2L}$, we <u>have</u> a <u>unique</u> solution for $\begin{cases} y' = f(x, y) \\ y(x_0) = y_0 \end{cases}$.

Numerical way:

$$\begin{cases} y' = f(x,y) \\ y(x_0) = y_0 \end{cases} \overset{\text{Picard}}{\Longleftrightarrow} y(x) = y_0 + \underline{\int_{x_0}^{x} f(t, y(t))\,dt}$$

Euler: $\underline{\int_{x_0}^{x} \underline{f(t, y(t))}dt \sim (x-x_0)\cdot f(x_0, y_0)}$

Heun: $\int_{x_0}^{x} f(t, y(t))\,dt \sim (x-x_0)\left(\frac{f(x_0,y_0) + f(x,y)}{2}\right)$

Classical Runge-Kutta:

$$\int_{x_0}^{x} f(t, y(t))dt \sim (x-x_0)\cdot\frac{f(x_0,y_0) + 4f(\overset{\text{middle point}}{\downarrow}) + f(x,y)}{6}$$

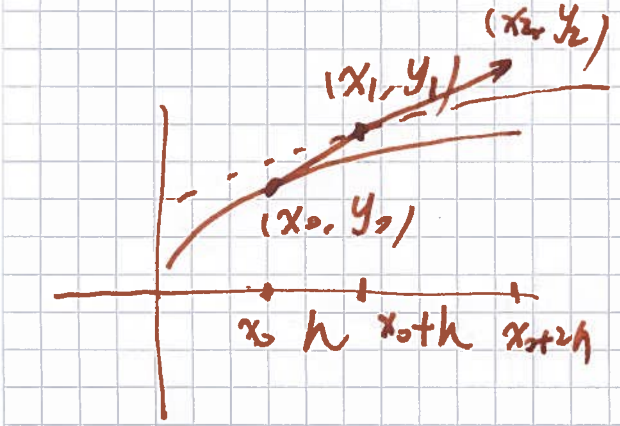
$$\uparrow$$

$$\int_0^1 f \sim \frac{f(0) + 4f(\frac{1}{2}) + f(1)}{6}$$

⑦ Euler:

$y(x) \approx y_0 + (x-x_0)\, f(x_0, y_0)$

Let $h = x - x_0 \Rightarrow y(x_0+h) \approx y_0 + h\, f(x_0, y_0)$



Start from (x_0, y_0)

$y_1 = y_0 + h\, f(x_0, y_0)$

$x_1 = x_0 + h$

$\vdots$

$$\begin{cases} y_{k+1} = y_k + h\, f(x_k, y_k) \\ x_{k+1} = x_k + h \end{cases}$$

Example:

$y' = -100y \qquad c \cdot e^{-100x}$ solution

$y_{k+1} = y_k + h \cdot (-100) y_k = (1 - h 100) y_k$

$|1 - h \cdot 100| > 1 \Rightarrow |y_{k+1}| > |y_k|$