Interpolation error Theorem:

$$\boxed{\begin{array}{l} \text{Given } f \in C^{n+1}[a,b] \\ n+1 \text{ distinct nodes } x_j \in [a,b] \end{array}}$$

Take p_n ($\deg = n$) s.t. $p_n(x_j) = f(x_j)$ $0 \le j \le n$.

Then for any $x \in [a,b]$, $\exists\ \xi(x) \in (a,b)$

$$\underline{f(x) - p_n(x)} = \frac{f^{(n+1)}(\xi(x))}{(n+1)!}(x-x_0)\cdots(x-x_n)$$

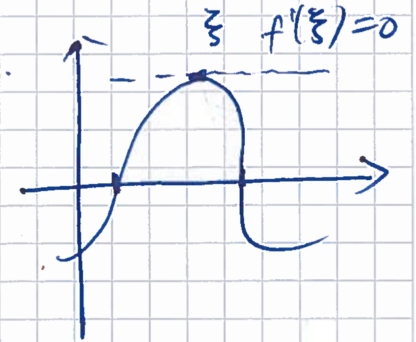
Error $:= e(x)$

Proof: Put $w(x) = (x-x_0)\cdots(x-x_n)$. Fix $x \in [a,b]$ $x \neq x_j$ $0 \le j \le n$.

Consider

$$\phi(t) := e(t)\,w(x) - e(x)\,w(t)$$

$\phi(t)$ has $n+2$ distint zeros : nodes + x.



$\phi'(\xi_1) = 0$ $\phi'(\xi_2) = 0$ $\phi'(\xi_{n+1}) = 0$

1 2 3 ⋯ $n+1$ $n+2$

⋮

$\phi''(\eta_1)$ ⋯ $\phi''(\eta_n) = 0$

$\Downarrow$ $\exists\ \xi(x) \in (a,b)$

$\phi^{(n+1)}(\xi(x)) = 0$

$\parallel$

$$f^{(n+1)}(\xi(x))\,w(x) - e(x)\cdot(n+1)! \Rightarrow e(x) = \frac{f^{(n+1)}(\xi(x))}{(n+1)!}w(x)$$

※

Equidistributed nodes: $x_j := a + jh$, $h = \dfrac{b-a}{n}$.

h h

x_0 x_1 x_2 ⋯ x_n

$a, b > 0$

$ab \le \left(\frac{a+b}{2}\right)^2$

If $x \in (x_j, x_{j+1})$ then

$$|(x-x_j)(x-x_{j+1})| = (x_{j+1}-x)(x-x_j) \le \left(\frac{x_{j+1}-x_j}{2}\right)^2 = \frac{h^2}{4}$$

$|x - x_{j-1}| \le 2h$ | $|x - x_{j+2}| \le 2h \le (j+2)h$

$|x - x_0| \le (j+1)h$ | $|x - x_n| \le nh$

$\Rightarrow |w(x)| \le \frac{h^2}{4}\cdot(2\cdots n)\,h^{n+1}$

$\frac{h^{n+1}}{4} n!$

Fact: Equi. nodes

$$\frac{(b-a)^{n+1}}{4(n+1)\cdot n^{n+1}}\cdot M$$

$$\|$$

$$|e(x)| \le \frac{h^{n+1}}{4(n+1)} M \qquad M := \max_{x\in[a,b]} |f^{n+1}(x)|$$

$$\forall x \in [a,b]$$

Example: $f(x) = \sin(x) \quad [a,b] = [0, 2\pi]$

$$\Downarrow$$

$$\sup_{[0,2\pi]} |\sin^{(n+1)}(x)| = 1 = M, \quad h = \frac{2\pi}{n}$$

$$\Downarrow$$

$$|e_n(x)| \le \frac{1}{4(n+1)}\cdot\left(\frac{2\pi}{n}\right)^{n+1} \quad \forall\ 0 \le x \le 2\pi$$

$\boxed{\text{Optimal nodes}}$:

motivation $\begin{cases} M \ (\leadsto f:\ x) \\ |w(x)| := |x-x_0|\cdots|x-x_n| \end{cases}$

Hope $\max_{x\in[a,b]} |x-x_0|\cdots|x-x_n|$ small.

$$\Downarrow$$

Chebyshev nodes: optimal choice of nodes in $[-1,1]$

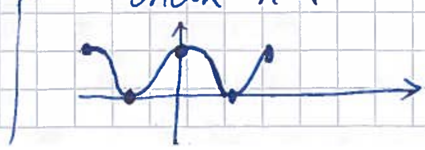
$$\tilde{x}_j := \cos\frac{(2j+1)\pi}{2(n+1)} \qquad j = 0,\cdots,n \ \sim W_{cheb}$$

Example: $n=1$. $\begin{cases} \tilde{x}_0 = \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2} \\ \tilde{x}_1 = \cos\frac{3\pi}{4} = -\frac{\sqrt{2}}{2} \end{cases}$ $\quad w(x) = x^2 - \frac{1}{2}$

$n=2$ $\begin{cases} \tilde{x}_0 = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2} \\ \tilde{x}_1 = \cos\frac{3\pi}{6} = 0 \\ \tilde{x}_2 = \cos\frac{5\pi}{6} = -\frac{\sqrt{3}}{2} \end{cases}$ $\quad w(x) = x\left(x^2 - \frac{3}{4}\right)$

$$\max_{x\in[-1,1]} |W_{cheb}(x)| = \frac{1}{2^n}$$

Check $n=1$



Key : (optional)

$$(x-\bar{x}_0)\cdots(x-\bar{x}_n) = \frac{1}{2^n}\cos\left((n+1)\cos^{-1}x\right)$$

Proof: When $x = \bar{x}_j = \cos\frac{(2j+1)\pi}{2(n+1)}$, we have

$$\cos(n+1)\cos^{-1}x = \cos\frac{(2j+1)\pi}{2} \equiv 0$$

It suffices to show $\forall\, n \geq 0.$

$$\underline{\frac{1}{2^n}\cos\left((n+1)\cos^{-1}x\right) = x^{n+1}+\cdots := T_{n+1}}$$

$n=-1 \quad T_0 \equiv 2$

Induction : $n=0$ ✓ $T_1 = \underline{x = x}$ ✓

Assum $n, n-1, \cdots 0$

$\Downarrow$?

$\underline{n+1}$.

$\cos(a+b)+\cos(a-b)$

$\cos(a+b) \quad = 2\cos a\cos b$

$= \cos a\cos b - \sin a\sin b$

$$\cos\left(n\cos^{-1}x + \cos^{-1}x\right) = 2^n T_{n+1}$$

$$\cos(n\cos^{-1}x)\cdot x - \sin(n\cos^{-1}x)\,\sin\cos^{-1}x$$

$$\cos\left(n\cos^{-1}x - \cos^{-1}x\right) = 2^{n-2}T_{n-1}$$

$\Rightarrow$

$$2^n T_{n+1} + 2^{n-2}T_{n-1} = 2\cos(n\cos^{-1}x)\cdot x$$

$$\Downarrow \qquad = 2\cdot 2^{n-1}\,T_n\cdot x$$

$$T_{n+1} = T_n\cdot x - \frac{1}{4}T_{n-1} = x(x^n+\cdots) - \frac{1}{4}(x^{n-1}+\cdots)$$

$$= x^{n+1}+\cdots$$ ✓

$\Downarrow$

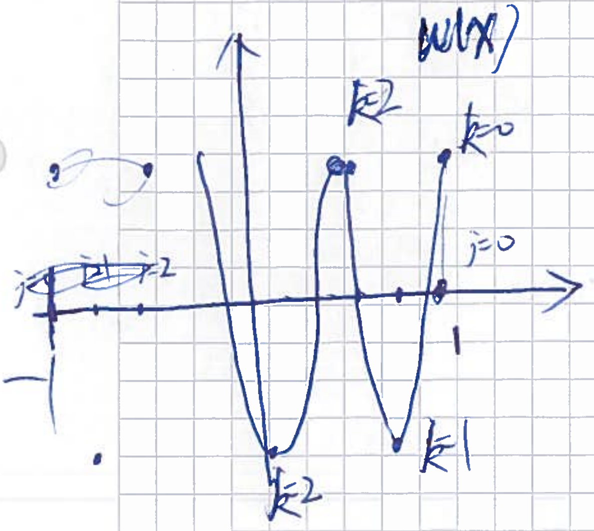
$$T_2 = T_1\cdot x - \frac{1}{4}\cdot 2 = x^2 - \frac{1}{2}$$

$$T_3 = \left(x^2-\frac{1}{2}\right)x - \frac{1}{4}\cdot x = x^3 - \frac{3}{4}x$$

Now : $|\cos| \leq 1 \Rightarrow |T_{n+1}| \leq \frac{1}{2^n}$,

$$T_{n+1} = \frac{1}{2^n} \Leftrightarrow (n+1)\cos^{-1}x = 2j\pi \Leftrightarrow x = \cos\frac{2j\pi}{n+1}$$

$$T_{n+1} = \frac{-1}{2^n} \Leftrightarrow (n+1)\cos^{-1}x = (2j+1)\pi \Leftrightarrow x = \cos\frac{(2j+1)\pi}{n+1}$$



$1 \cdot \int \frac{k\pi}{n+1}$ $\qquad$ $0 \le k \le n+1$

Assume

$$S(x) = x^{n+1} + \cdots +$$

sup

$$\max_{x\in[-1,1]} |S(x)| < 1/2^n$$

Then

$(T_{n+1} - S)$ has $(n+1)$ zero points

$\neq 0$

$\deg(T_{n+1} - S) \le n$

Impossible #

$n+1$, $k=\ldots$ $\quad \cdots$ $k=2$ $\quad k=1$ $\quad k=0$

>0 $\quad <0$ $\quad >0$

ξ_2 $\quad \xi_1$ zero points

Numerical integration

Aim: Compute $\int_a^b f(x)\,dx := I$

Idea: Find Polynomial $p_n \sim f$ then

$$Q := \underbrace{\int_a^b p_n\,dx}_{\text{computable.}} \sim I$$

Example: Lagrange ~~polynomial~~ (Given points $x_0, \cdots x_n$) nodes → NO.

$$p_n = f(x_0)\,l_0 + \cdots + f(x_n)\,l_n$$

$\Downarrow$

$$Q = \int_a^b f(x_0)\,l_0 + \cdots + f(x_n)\,l_n$$

$$= \sum f(x_i) \int_a^b l_i$$

Call $w_i := \int_a^b l_i$: weights!