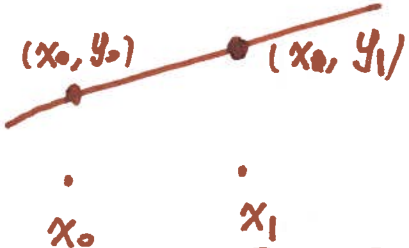
Polynomial interpolation

Problem: Find

$$p(x) = c_0 + c_1 x + \cdots + c_n x^n$$

: $p(x_j) = y_j$, $0 \le j \le n$



1. Direct
2. Lagrange
3. Newton

x_j : nodes

1. Direct approach

$$c_0 + c_1 x_j + \cdots + c_n x_j^n = y_j \qquad 0 \le j \le n$$

Vander-monde matrix

$$\begin{pmatrix} 1 & x_0 & \cdots & x_0^n \\ 1 & x_1 & \cdots & x_1^n \\ & \vdots & & \\ 1 & x_n & \cdots & x_n^n \end{pmatrix} \begin{pmatrix} c_0 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} y_0 \\ \vdots \\ y_n \end{pmatrix}$$

$$AC = Y \iff C = A^{-1}Y$$

known; A^{-1} exists; Hard to compute

$$x_j \ne x_k \ (j \ne k)$$
$$\Downarrow$$
$$\det A \ne 0$$

$$\det A = (x_n - x_0) \cdots (x_n - x_{n-1})$$
$$(x_{n-1} - x_0) \cdots (x_{n-1} - x_{n-2})$$
$$\cdots$$
$$(x_1 - x_0)$$
$$= \prod_{0 \le j < k \le n} (x_k - x_j)$$

Example: $\det\begin{pmatrix} 1 & x_0 \\ 1 & x_1 \end{pmatrix} = x_1 - x_0$

$$\det\begin{pmatrix} 1 & x_0 & x_0^2 \\ 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \end{pmatrix} = x_1 x_2 (x_2 - x_1) - x_0 (x_2^2 - x_1^2) + x_0^2 (x_2 - x_1)$$
$$= (x_2 - x_1)\underline{(x_1 x_2 - x_0 (x_2 + x_1) + x_0^2)}$$
$$= (x_2 - x_1)\underline{(x_2 - x_0)(x_1 - x_0)}$$

2. Lagrange

Idea:

Remove $(x-x_j)$

$$l_j(x) = \frac{(x-x_0)\cdots(x-x_{j-1})(x-x_{j+1})\cdots(x-x_n)}{(x_j-x_0)\cdots(x_j-x_{j-1})(x_j-x_{j+1})\cdots(x_j-x_n)}$$

$$l_j(x) = \begin{cases} 0 & x = x_k,\ k \neq j \\ 1 & x = x_j \end{cases} \Rightarrow$$

→ Cardinal functions

Define

$$p(x) = y_0 l_0(x) + \cdots + y_n l_n(x) \quad (*)$$

satisfies

$$p(x_k) = \underbrace{y_0 l_0(x_k)}_{0} + \cdots + \underbrace{y_k l_k(x_k)}_{y_k} + \cdots + \underbrace{y_n l_n(x_k)}_{0} = y_k$$

Example:

x_j	0	1	3
y_j	3	8	6

$n=2$

lagrange $\Rightarrow$

$$l_0 = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} = \frac{(x-1)(x-3)}{(0-1)(0-3)} = \frac{x^2-4x+3}{3}$$

$$l_1 = \frac{(x-0)(x-3)}{(1-0)(1-3)} = \frac{x^2-3x}{-2}$$

$$l_2 = \frac{(x-0)(x-1)}{(3-0)(3-1)} = \frac{x^2-x}{6}$$

$\Downarrow (*)$

$$p(x) = 3l_0 + 8l_1 + 6l_2 = x^2-4x+3 + (-4)(x^2-3x) + x^2-x$$

$$= -2x^2+7x+3 \quad \#$$

3. Newton interpolation

Idea: consider Newton form

$$p(x) = \underline{c_n(x-x_0)\cdots(x-x_{n-1})} + \cdots + \underline{c_2(x-x_0)(x-x_1)} + c_1(x-x_0) + c_0$$

satisfies (Hope) $\left\{ \begin{array}{l} p(x_0) = y_0 = f(x_0) \\ \vdots \\ p(x_n) = y_n = f(x_n) \end{array} \right.$ what are c_n's?

Example: (1) $p(x_0) = 0 + \cdots 0 + c_0 \Rightarrow \underline{c_0 = f(x_0)}$

(2) $p(x_1) = c_1(x_1 - x_0) + f(x_0) = f(x_1)$

$$\Downarrow$$

$$c_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

Def: 0-th order finite difference : $f[x_0] = f(x_0)$

1-th ------ $f[x_0, x_1] := \dfrac{f(x_1) - f(x_0)}{x_1 - x_0} = \dfrac{f[x_1] - f[x_0]}{x_1 - x_0}$

$\vdots$

k-th -- $f[x_0, \cdots, x_k] := \dfrac{f[x_1, \cdots, x_k] - f[x_0, \cdots, x_{k-1}]}{x_k - x_0}$

Theorem: $c_k = f[x_0, \cdots, x_k]$

Proof: Induction on k.

$k=0$ ✓

$k=1$ ✓

Assume that $c_{n-1} = f[x_0, \cdots, x_{n-1}]$:

$$\Downarrow$$

$$\begin{cases} p_{x_0 \cdots x_{n-1}} = f[x_0, \cdots, x_{n-1}]\, x^{n-1} + \cdots \\ p_{x_1 \cdots x_n} = f[x_1, \cdots, x_n]\, x^{n-1} + \cdots \end{cases}$$

$$\left.\begin{matrix} x_0\ x_1 \cdots x_{n-1} \\ x_1 \cdots \cancel{x_{n-1}}\ x_n \end{matrix}\right\} \quad p_{x_0 \cdots x_{n-1}}(x_j) = p_{x_1 \cdots x_n}(x_j) = f(x_j)$$

$$\Downarrow \ \deg = n-1 \qquad \underline{1 \le j \le n-1}.$$

$$p_{x_0 \cdots x_{n-1}} - p_{x_1 \cdots x_n} = \Big(f[x_0, \cdots, x_{n-1}] - f[x_1, \cdots, x_n]\Big)$$

$$= c \cdot (x - x_1) \cdots (x - x_{n-1})$$

$$p_{x_0 \cdots x_n} = C_n x^n + \cdots$$

$$p_{x_0 \cdots x_n}(x_j) = p_{x_0 \cdots x_{n-1}}(x_j) \quad 0 \le j \le n-1$$

$$\Downarrow$$

$$p_{x_0 \cdots x_n} - p_{x_0 \cdots x_{n-1}} = C_n (x-x_0)\cdots(x-x_{n-1}) \quad ①$$

similarly

$$p_{x_0 \cdots x_n} - p_{x_1 \cdots x_n} = C_n (x-x_1)\cdots(x-x_n) \quad ②$$

$\Rightarrow$

$$(② - ① =) \; p_{x_0 \cdots x_{n-1}} - p_{x_1 \cdots x_n}$$

$$= C_n (x-x_1)\cdots(x-x_{n-1}) \big((x-x_n) - (x-x_0) \big)$$

$$= C_n (x-x_1)\cdots(x-x_{n-1})(x_0 - x_n)$$

$$\big(f[x_0, \ldots, x_{n-1}] - f[x_1, \ldots, x_n] \big) = C_n (x_0 - x_n)$$

$$\Downarrow$$

$$C_n = \frac{f[x_1, \ldots, x_n] - f[x_0, \ldots, x_{n-1}]}{x_n - x_0} = f[x_0, \ldots, x_n] \quad \#$$

Example (Newton interpolation)

$$
\begin{array}{c|lll}
0 & 1 & & \\
 & & \dfrac{1-\frac{1}{2}}{0-\frac{2}{3}} = \dfrac{\frac{1}{2}}{-\frac{2}{3}} = \boxed{-\frac{3}{4}}. & \\
\frac{2}{3} & \frac{1}{2} & & \dfrac{-\frac{3}{4}-\left(-\frac{3}{2}\right)}{0-1} = -\dfrac{3}{4} \\
 & & \dfrac{\frac{1}{2}-0}{\frac{2}{3}-1} = \dfrac{\frac{1}{2}}{-\frac{1}{3}} = -\dfrac{3}{2}. & \\
1 & 0 & &
\end{array}
$$

$$p(x) = -\frac{3}{4}(x-0)\left(x-\frac{2}{3}\right) + \boxed{-\frac{3}{4}}(x-0) + 1$$

$$= -\frac{3}{4}x^2 - \frac{1}{4}x + 1.$$