① **Recall**:

Euler : $\int_0^1 f \sim f(0)$

Heun : $\int_0^1 f \sim \frac{1}{2}(f(0)+f(1))$

Classical Runge-Kutta : $\int_0^1 f \sim \frac{1}{6}(f(0)+4f(\frac{1}{2})+f(1))$

$\longrightarrow \begin{cases} y'(x) = f(x, y(x)) & (*) \\ y(x_0) = y_0 \end{cases}$

$\Updownarrow$ integrate $(*)$

$y(x) - y_0 = \int_{x_0}^{x} f(t, y(t))\,dt \quad (**)$

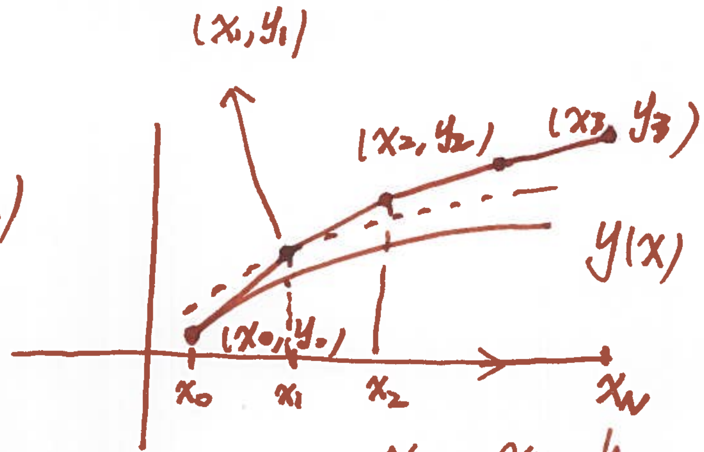
Euler: $\int_{x_0}^{x} f(t, y(t))\,dt \sim (x-x_0)\cdot f(t, y(t))\Big|_{t=x_0} = (x-x_0)\, f(x_0, y_0)$

$x_1 = x_0 + h$

want to find $y_1 \sim y(x_1)$

$y_1 - y_0 \sim h\, f(x_0, y_0)$

Euler: $\begin{cases} y_{n+1} = y_n + h\, f(x_n, y_n) \\ x_{n+1} = x_n + h \end{cases}$



$x_{j+1} - x_j = h$

Def. Solve ODE over $[x_0, x_N]$

$|e_N| \triangleq |y(x_N) - y_N| \leq C\cdot h^p \qquad h = \frac{x_N - x_0}{N}$

$\Downarrow$ Method is of order p

Example:

Euler : order 1

Heun : order 2

order ~ degree of precision +1

Control local error:

(2)

$$x_1 = x_0 + h$$

Aim: Find $y_1 \sim y(x_1)$

Method 1: $\cancel{y(x_1)} - \underline{y_1} = D \cdot h^{p+1} + \underline{d \cdot h^{p+2} + \cdots}$

Method 2: $\cancel{y(x_1)} - \underline{\hat{y}_1} = 0 + \hat{D}\, h^{p+2} + \cdots$

$$M1 - M2 = \underline{\hat{y}_1 - y_1} = \underline{D \cdot h^{p+1}} + \cdots \sim \underline{y(x_1) - y_1}$$

$\downarrow$ local error estimate (for the real error)

Example:

$$\begin{cases} y' = -2xy \\ y(0) = 1 \end{cases} \quad h = 0.1$$

Euler $\underline{\overset{\text{order}}{(1)}}$: $y_1 = y_0 + h f(x_0, y_0)$

$= 1 + 0.1 \cdot (-2) \cdot 0 \cdot 1$

$= 1$

Heun: order (2): $K_1 = f(x_0, y_0)$ | $K_1 = 0$

$K_2 = f(x_0 + h, \underline{y_0 + hK_1})$ | $K_2 = -2 \cdot 0.1 = -0.2$

$\hat{y}_1 = 1 + \frac{0.1}{2}(0 - 0.2) \Leftarrow \hat{y}_1 = y_0 + \frac{h}{2}(K_1 + K_2)$

$= 0.99$

$\underline{\text{local error estimate}}$: $\underline{\hat{y}_1 - y_1} = -0.01$

$\underline{\text{real error}}$: $y(x_1) - y_1 = e^{-0.1^2} - 1$

$y(x) = e^{-x^2}$

$\leq -0.01 + [illegible] \times 10^{-4}$

Step size control

3

Fix : Tol : tolerance $\quad c \cdot 10^{-4}$

If $|\hat{y}_1 - y_1| \leq Tol$

increase h

else $|\hat{y}_1 - y_1| \geq Tol$

decrease h

General Runge-Kutta:

Example (II : $\boxed{2}$)

$$\begin{cases} y' = f(y) \\ y(x_0) = y_0 \end{cases} \quad \Big| \quad f: \text{only depends on } y$$

Method

$$\left\{ \begin{aligned} k_1 &= f(y_n) \\ k_2 &= f(y_n + h a_{21} k_1) \\ y_{n+1} &= y_n + h \cdot (b_1 k_1 + b_2 k_2) \end{aligned} \right\} \xRightarrow{n=0} \left\{ \begin{aligned} k_1 &= f(y_0) \\ k_2 &= f(y_0 + h a_{21} f(y_0)) \\ y_1 &= y_0 + h \cdot b_1 f(y_0) + h b_2 \cdot k_2 \end{aligned} \right.$$

④

$$k_2 = f(y_0) + h a_{21} f(y_0) f'(y_0) + O(h^2)$$

$$\Downarrow$$

$$y_1 = \cancel{y_0} + \underline{h b_1 f(y_0)} + h b_2 \left(f(y_0) + h a_{21} f(y_0) f'(y_0) + \underline{o(h^2)} \right)$$

real value:

$$y(x_0+h) = \cancel{y(x_0)} + \underline{h\, y'(x_0)} + \underline{\frac{h^2}{2} y''(x_0)} + o(h^3)$$

Hope that $y_1 \sim y(x_0+h)$

$$\Downarrow$$

$$y_1 - y(x_0+h) = h\, y'(x_0)(b_1 - 1 + b_2) + h^2 b_2 a_{21} \underbrace{f(y_0) f'(y_0)}_{=\, y''(x_0)} - h^2 \cdot \underline{\frac{y''(x_0)}{2}} + o(h^3)$$

$$y'(x) = f(y(x))$$

$$\Downarrow$$

$$(*) \quad \underline{y''(x) = f'(y(x))\, y'(x)}$$

$$\Rightarrow \quad f(y_0) f'(y_0) = y'(x_0) f'(y(x_0)) \overset{(*)}{=} y''(x_0)$$

Summary: $y_1 - y(x_0+h)$

$$= h\, y'(x_0)(b_1 + b_2 - 1) + h^2 \left(b_2 a_{21} - \frac{1}{2} \right) y''(x_0) + o(h^3)$$

$$\left\{ \begin{array}{l} b_1 + b_2 = 1 \\ b_2 a_{21} = \frac{1}{2} \end{array} \right.$$

$$\Downarrow$$

$$y_1 - y(x_0+h) = o(h^3)$$

5

Stability analysis: (for stiff ODEs)

Idea: $\begin{cases} y' = \lambda y \\ y(0) = y_0 \end{cases}$ λ: real constant.

$f(x, y) = \lambda y$

Euler method: $y_1 = y_0 + h f(x_0, y_0)$

$= y_0 + h \lambda \cdot y_0$

$= (1 + h\lambda) y_0$

$\vdots$

$y_{n+1} = y_n + h(f(x_n, y_n))$

$= y_n + h\lambda y_n$

$= (1 + h\lambda) y_n$

$= (1 + h\lambda)(1 + h\lambda) y_{n-1}$

$= \cdots = (1 + h\lambda)^{n+1} y_0$

$= (1 - 10^4 \cdot h)^{n+1} \cdot 1$

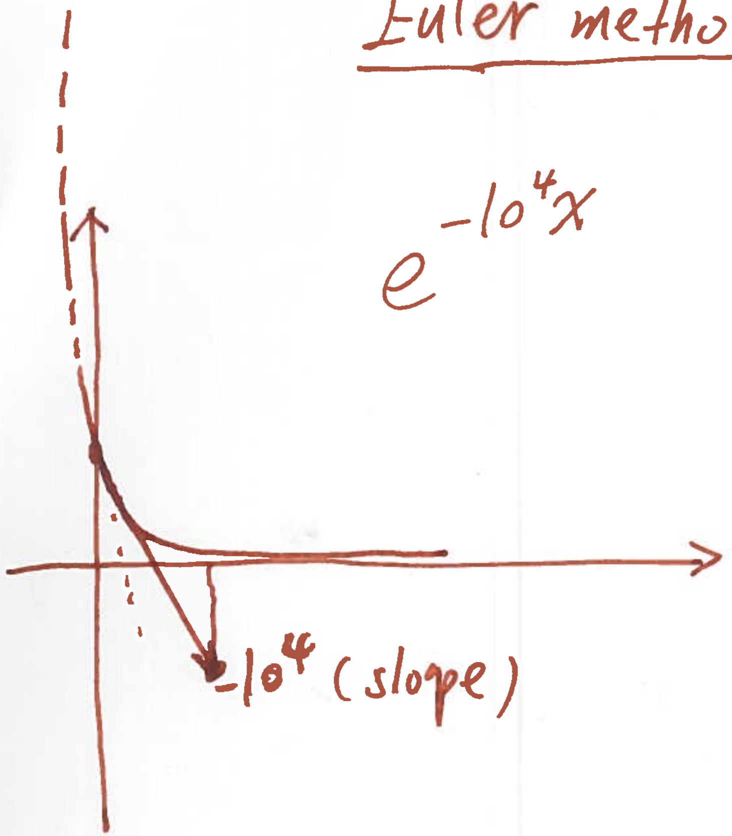
$y = e^{\lambda x} \cdot y_0$ solution

$\begin{cases} \lambda = -10^4 \\ y_0 = 1 \end{cases}$

$y = e^{-10^4 x}$

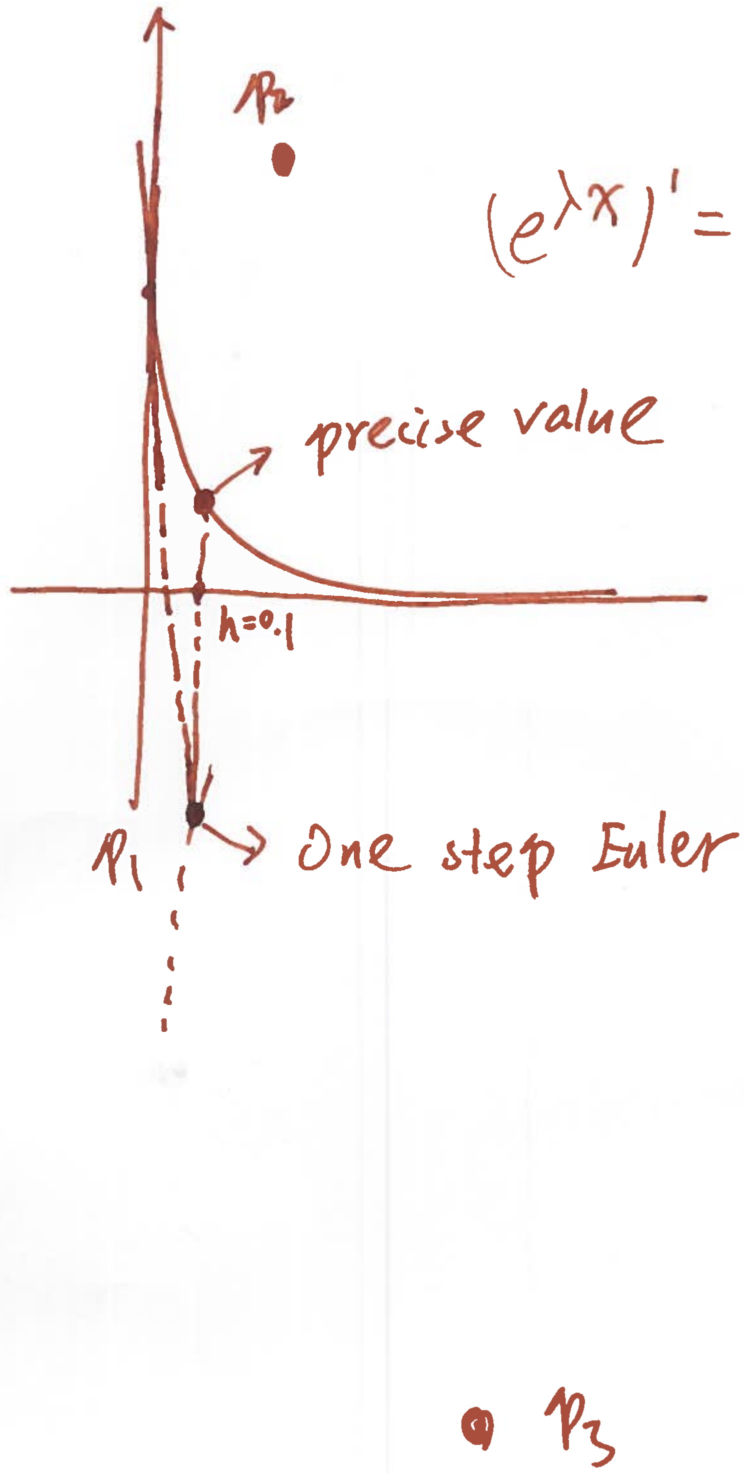
Even when $h = 10^{-3}$ $y_{n+1} = (-9)^{n+1}$

$e^{-10^4 x}$



⑥

$$(e^{\lambda x})' = \lambda e^{\lambda x}\Big|_{x=0} = \lambda$$



<u>Def</u>: Stiff ODE

$y' = \lambda y$ stiff on $(0, \infty)$

if

$-\lambda$ is <u>large enough</u>!