Norwegian University of
Science and Technology

Department of Mathematical Sciences

Examination paper for **TMA4125 Matematikk 4N**

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Permitted examination support material: Kode C:
Bestemt, enkel kalkulator
Rottmann: Matematisk formelsamling

Other information:

- All answers have to be justified, and they should include enough details in order to see how they have been obtained.
- All sub-problems carry the same weight for grading.
- Good Luck!

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Checked by:

Informasjon om trykking av eksamensoppgave	
Originalen er:	
1-sidig ☐	**2-sidig** ☒
sort/hvit ☒	**farger** ☐
skal ha flervalgskjema ☐	

Date Signature

Problem 1 Let $f(x)$ be defined as $f(x) = x$ for $x \in [0,3]$.

a) Find the Fourier sine series of $f(x)$.

b) Use the result to compute the value of the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$.

Hint: Use Parseval's identity.

Solution: **a)** The Fourier sine series is of the form

$$\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right),$$

where $L = 3$ is the half-range period, and

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right) \mathrm{d}x.$$

We compute the coefficients b_n:

$$b_n = \frac{2}{3}\int_0^3 x\sin\left(\frac{n\pi x}{3}\right) \mathrm{d}x.$$

To compute this integral, we use integration by parts. We obtain

$$\begin{aligned} b_n &= \frac{2}{3}\left(\left[x\frac{-3}{n\pi}\cos\left(\frac{n\pi x}{3}\right)\right]_0^3 + \frac{3}{n\pi}\int_0^3 \cos\left(\frac{n\pi x}{3}\right) \mathrm{d}x\right) \\ &= \frac{2}{3}\left(\frac{(-1)^{n+1}\cdot 9}{n\pi} + 0\right) = \frac{(-1)^{n+1}\cdot 6}{n\pi}. \end{aligned}$$

Thus the Fourier sine series of $f(x)$ is given by

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}\cdot 6}{n\pi}\sin\left(\frac{n\pi x}{3}\right).$$

b) We use Parseval's identity:

$$\sum_{n=1}^{\infty} b_n^2 = \frac{2}{L}\int_0^L f(x)^2 \mathrm{d}x.$$

We have

$$\frac{36}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{2}{3}\int_0^3 x^2 \mathrm{d}x = 6.$$

Thus

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Problem 2 Let

$$f(x) = \frac{4}{x^2+2} \qquad \text{and} \qquad g(x) = \frac{2x}{(x^2+1)^2} \qquad \text{for all real } x.$$

Show that the convolution

$$(f * g)(x) = -\pi\sqrt{2}i \int_{-\infty}^{\infty} \omega e^{-(\sqrt{2}+1)|\omega|} e^{i\omega x} \mathrm{d}\omega.$$

Solution: We first compute $\mathcal{F}(f * g)$, the Fourier transform of the convolution. By the convolution theorem, we get

$$\mathcal{F}(f * g) = \sqrt{2\pi}\mathcal{F}(f)\mathcal{F}(g).$$

Using the tables, we see that

$$\mathcal{F}(f) = 2\sqrt{\pi}e^{-\sqrt{2}|\omega|}.$$

We note that

$$g(x) = -\left(\frac{1}{x^2+1}\right)'.$$

Using the properties of the tables, we obtain

$$\mathcal{F}(g) = -i\omega\sqrt{\frac{\pi}{2}}e^{-|\omega|}.$$

Thus,

$$\mathcal{F}(f * g) = -2i\pi\sqrt{\pi}\omega e^{-(\sqrt{2}+1)|\omega|}.$$

Applying the inverse Fourier transform on both sides, we obtain

$$(f * g)(x) = -\sqrt{2}\pi i \int_{-\infty}^{\infty} \omega e^{-(\sqrt{2}+1)|\omega|} e^{i\omega x} \mathrm{d}\omega.$$

Problem 3 Using Laplace transforms, solve the differential equation

$$y'' + 3y' + 2y = \begin{cases} 4t & \text{if } 0 < t \leq 1 \\ 4 & \text{if } t > 1, \end{cases}$$

with initial conditions $y(0) = 0$ and $y'(0) = 0$.

Solution: We note that the differential equation can be written as

$$y'' + 3y' + 2y = 4t(1 - u(t-1)) + 4u(t-1) = 4t - 4(t-1)u(t-1).$$

Applying the Laplace transform on both sides and isolating $\mathcal{L}(y)$, we obtain

$$\mathcal{L}(y) = \frac{4}{s^2(s+1)(s+2)} - e^{-s}\left(\frac{4}{s^2(s+1)(s+2)}\right).$$

We write the rational fraction as partial sums:

$$\frac{4}{s^2(s+1)(s+2)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+1} + \frac{D}{s+2}$$

and find $A = -3$, $B = 2$, $C = 4$ and $D = -1$. Applying the inverse Laplace transform, we obtain

$$y(t) = (-3 + 2t + 4e^{-t} - e^{-2t}) - (-3 + 2(t-1) + 4e^{-(t-1)} - e^{-2(t-1)})u(t-1).$$

Problem 4 Consider the equation $x = \frac{1}{2}\cos(x)$.

a) Show that this equation has a unique solution in the interval $(0, 1)$.

b) Compute 2 iterations of Newton's method to approximate the solution, starting with $x_0 = 0.5$.

Keep 5 digits in your answers.

Solution:

a) Apply the fixed point theorem on $x = g(x) = \dfrac{1}{2}\cos(x)$.

- $|g'(x)| = \dfrac{1}{2}|\sin(x)| \leq \dfrac{1}{2} < 1.$
- The function $\frac{1}{2}\cos(x)$ is monotonically decreasing on the interval $[0, 1]$, thus $g(x) \in (g(1), g(0)) = (0.27015, 0.5) \in (0, 1)$

so the two conditions of the fixed point theorem are fulfilled, and the fixed exist and is unique.

b) Rewrite the equation to the form $f(x) = x - \frac{1}{2}\cos(x) = 0$ and Newtons method becomes

$$x_{k+1} = x_k - \frac{x_k - \frac{1}{2}\cos(x_k)}{1 + \frac{1}{2}\sin(x_k)} = \frac{x_k \sin(x_k) + \cos(x_k)}{2 + \sin(x_k)}.$$

and the first iterations becomes

$$x_0 = 0.50000, \qquad x_1 = 0.45063, \qquad x_2 = .45018.$$

Problem 5 Given the differential equation

$$y' = xy^2, \qquad y(1) = 0.5.$$

a) Compute the approximate solution $y_1^H \approx y(1.1)$ by one step of the Heun method with step size $h = 0.1$.

b) Compute the approximate solution $y_1^E \approx y(1.1)$ by one step of the Euler method with step size $h = 0.1$.

Use the result from point **a)** to find an estimate for the error $y(1.1) - y_1^E$.

Given a user specified tolerance $\text{Tol} = 10^{-3}$, will you accept y_1^E as a sufficient accurate solution?

Whether you accept the step or not, what should your next step size be? Use $P = 0.8$ as the pessimist factor in the step size selection algorithm.

Hint: The Euler method is of order 1, the Heun method is of order 2.

Solution:

a) One step of Heuns method with $h = 0.1$ becomes

$$y_1^* = y_0 + hx_0y_0^2 = 0.5 + 0.1 \cdot 1 \cdot 0.5^2 = 0.525$$
$$y_1^H = y_0 + \frac{h}{2}\Big(x_0y_0^2 + x_1(y_1^*)^2\Big) = 0.5 + 0.5 \cdot 0.1 \cdot \big(1 \cdot 0.5^2 + 1.1 \cdot 0.525^2\big) = 0.52766.$$

b)

- One step of Eulers method: $y_1^E = y_1^* = 0.525$.

- Error estimate: $y(1.1) - y_1^E \approx le_1 = y_1^H - y_1^E = 2.66 \cdot 10^{-3}$.
- The step is not accepted, and will be repeated with the new stepsize

$$h_{new} = P\left(\frac{\text{Tol}}{|le_1|}\right)^{\frac{1}{p+1}} h_{old} = 0.8\sqrt{\frac{10^{-3}}{2.66 \cdot 10^{-3}}} \cdot 0.1 = 0.049.$$

Here, $p = 1$ is the order of the Euler method.

Problem 6

a) For a given function $f(t)$, give an expression for the polynomial of lowest possible degree interpolating the function in the nodes $t_0 = -1/3$ and $t_1 = 1$.

Use the result to find a quadrature rule $Q[-1,1] = w_0 f(t_0) + w_1 f(t_1)$ as an approximation to the integral $\int_{-1}^{1} f(t)\mathrm{d}t$.

Find the degree of precision of the quadrature rule.

b) Transfer the quadrature rule $Q[-1,1]$ to some arbitrary interval $[a,b]$.

Use it to find an approximation of the integral $\int_1^2 x^2 \sin(\pi x/2)\mathrm{d}x$.

Solution:

a) Using Lagrange interpolation, we get

$$p_1(t) = f(t_0)\frac{t-t_1}{t_0-t_1} + f(t_1)\frac{t-t_0}{t_1-t_0} = \frac{3}{4}\Big(f(t_0)(1-t) + f(t_1)(t+\frac{1}{3})\Big).$$

The quadrature rule is found by integrating this polynomial:

$$Q[-1,1] = \int_{-1}^{1} p_1 \mathrm{d}t = \frac{3}{2}f(t_0) + \frac{1}{2}f(t_1).$$

The quadrature Q has degree of precision d if $Q[-1,1] = \int_{-1}^{1} f(t)\mathrm{d}t$ for all polynomials of degree d or less. That is, if this is true for all $f = t^l$, $l = 0, 1, \ldots, d$ but not for $l = d+1$.

$$
\begin{array}{lll}
f = 1 & \int_{-1}^{1} \mathrm{d}t = 2 & Q[-1,1] = \frac{3}{2}\cdot 1 + \frac{1}{2} = 2 \\
f = t & \int_{-1}^{1} t\mathrm{d}t = 0 & Q[-1,1] = \frac{3}{2}\cdot\left(-\frac{1}{3}\right) + \frac{1}{2}\cdot 1 = 0 \\
f = t^2 & \int_{-1}^{1} t^2\mathrm{d}t = \frac{2}{3} & Q[-1,1] = \frac{3}{2}\cdot\left(-\frac{1}{3}\right)^2 + \frac{1}{2}\cdot 1^2 = \frac{2}{3} \\
f = t^3 & \int_{-1}^{1} t^3\mathrm{d}t = 0 & Q[-1,1] = \frac{3}{2}\cdot\left(-\frac{1}{3}\right)^3 + \frac{1}{2}\cdot 1^3 = \frac{4}{9}
\end{array}
$$

so the quadrature rule has degree of precision 2.

(The rule will have at least degree of precision 1 by construction, so to check for $f = 1$ and $f = t$ is superfluous. On the other hand, it is useful for checking that the previous result was correct.)

b) Use the mapping $x = \frac{b-a}{2}t + \frac{b+a}{2}$. Then

$$\int_a^b f(x)\mathrm{d}x = \frac{b-a}{2}\int_{-1}^{1} f(\frac{b-a}{2}t + \frac{b+a}{2})\mathrm{d}t.$$

Using the nodes $x_i = \frac{b-a}{2}t_i + \frac{b+a}{2}$ for $i = 0, 1$, we get that

$$x_0 = \frac{2a}{3} + \frac{b}{3}, \qquad x_1 = b$$

and the quadrature formula becomes

$$\tilde{Q}[a,b] = \frac{b-a}{4}\left[3f\left(\frac{2}{3}a + \frac{b}{3}\right) + f(b)\right].$$

Applied to $\int_1^2 x^2 \sin(\pi x/2)\mathrm{d}x$ we get $x_0 = 4/3$ and $x_1 = 2$ and

$$\tilde{Q}[1,2] = \frac{1}{4}\cdot\left[3\cdot\left(\frac{4}{3}\right)^2\cdot\sin\left(\frac{\pi}{2}\cdot\frac{4}{3}\right) + 2^2\cdot\sin\left(\frac{\pi}{2}\cdot 2\right)\right] = \frac{2}{3}\sqrt{3} = 1.154700539.$$

(For comparision, the exact value of the integral is $1.219885070\ldots$).

Problem 7 Let $u(x,t)$ be the deflection at time t and position x of a vibrating string of length 4. It satisfies the wave equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \quad 0 \le x \le 4, \quad t \ge 0,$$

with initial conditions

$$u(x,0) = 3\sin(\pi x) \quad \text{and} \quad u_t(x,0) = \sin(4\pi x), \quad 0 \le x \le 4,$$

and boundary conditions

$$u(0,t) = u(4,t) = 0, \quad t \ge 0.$$

a) Find the solutions that are of the form $u(x,t) = F(x)G(t)$ and satisfy the boundary conditions.

b) Find the solution that satisfy the initial conditions.

c) The aim is now to set up a numerical scheme for the equation.

Use step sizes Δt and Δx in the $t-$ and $x-$direction respectively with $\Delta x = 4/M$, where M is the number of intervals in the $x-$direction. The gridpoints are then given by $t_n = n\Delta t$, $n = 0, 1, 2, \ldots$ and $x_i = i\Delta x$, $i = 0, 1, \ldots, M$ Let $U_i^n \approx u(x_i, t_n)$ be the numerical approximation in each gridpoint.

- Set up a finite difference scheme for the equation, based on central differences.
  The scheme will be explicit, in the sense that U_i^{n+1} can be expressed in terms of the numerical solutions at time steps t_n and t_{n-1} for $n \ge 1$.
- Use a central difference for $u_t(x,0)$ and the idea of a false boundary to find a scheme for computing U_i^1, $i = 1, \ldots, M-1$.
- Let $\Delta x = 0.1$ and $\Delta t = 0.1$ and use your schemes to find $U_1^2 \approx u(0.1, 0.2)$.

Solution: **a)** We plug $u(x,t) = F(x)G(t)$ into the wave equation and obtain two ODEs

$$F'' - kF = 0$$
$$G'' - kG = 0.$$

We split into three cases $k = 0$, $k > 0$ and $k < 0$. The only non-trivial solution is for $k = -p^2 < 0$. The first equation then have solution

$$F(x) = A\cos px + B\sin px.$$

From the boundary conditions, we get $F(0) = 0$ and $F(4) = 0$, so $A = 0$ and

$$\sin 4p = 0, \qquad \text{hence} \qquad p = \frac{n\pi}{4}, \qquad n = 1, 2, \dots$$

We can set $B = 1$. We now solve the second equation with $k = -p^2 = -(\frac{n\pi}{4})^2$. It has a solution of the form

$$G_n(t) = B_n \cos pt + B_n^* \sin pt.$$

The solutions of the form $F(x)G(t)$ are thus given by

$$u_n(x,t) = \left(B_n \cos\left(\frac{n\pi}{4}t\right) + B_n^* \sin\left(\frac{n\pi}{4}t\right)\right) \sin\left(\frac{n\pi}{4}x\right)$$

b) To find a solution that satisfies the initial condition, we need to sum over those functions. We have

$$u(x,t) = \sum_{n=1}^{\infty} \left(B_n \cos\left(\frac{n\pi}{4}t\right) + B_n^* \sin\left(\frac{n\pi}{4}t\right)\right) \sin\left(\frac{n\pi}{4}x\right).$$

We thus have, using the initial condition,

$$3\sin(\pi x) = u(x,0) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{4}.$$

This is a Fourier sine series. Comparing the two Fourier series, we see that the coefficients B_n are given by

$$B_4 = 3, \qquad B_n = 0, \quad \text{if } n \neq 4.$$

To obtain the coefficients B_n^*, we take the derivative of the series with respect to t:

$$u_t(x,t) = \sum_{n=1}^{\infty} \left(-B_n \frac{n\pi}{4} \sin\left(\frac{n\pi}{4}t\right) + B_n^* \frac{n\pi}{4} \cos\left(\frac{n\pi}{4}t\right)\right) \sin\left(\frac{n\pi}{4}x\right).$$

Using the other initial condition:

$$\sin(4\pi x) = u_t(x,0) = \frac{n\pi}{4} \sum_{n=1}^{\infty} B_n^* \sin\left(\frac{n\pi}{4}x\right).$$

Comparing again the two Fourier series, we have

$$B_{16}^* = \frac{1}{4\pi}, \qquad B_n^* = 0, \quad \text{if } n \neq 16.$$

Thus, the solution to the differential equation is

$$u(x,t) = 3\cos(\pi t)\sin(\pi x) + \frac{1}{4\pi}\sin(4\pi t)\sin(4\pi x).$$

c)

- The corresponding difference equations for the point (x_i, t_n) using central differences becomes

$$\frac{U_i^{n+1} - 2U_i^n + U_i^{n-1}}{\Delta t^2} = \frac{U_{i+1}^n - 2U_i^n + U_{i-1}^n}{\Delta x^2}.$$

  Solve this with respect to U_i^{n+1}, set $r = \Delta x/\Delta t$ for simplicity (not required), and the algorithm becomes: For $n = 1, 2, \dots$:

$$\begin{aligned} &U_i^{n+1} = r^2(U_{i+1}^n + U_{i-1}^n) + 2(1-r^2)U_i^n - U_i^{n-1}, \qquad i = 1, \dots, M-1, \quad (1) \\ &U_0^{n+1} = 0, \qquad U_M^{n+1} = 0. \end{aligned}$$

- Extend the solution domain to $t = -\Delta t$, and use a central difference formula to approximate the starting condition $u_t(x, 0) = \sin(4\pi x)$. For $n = 0$ we have the following two equations:

$$\begin{aligned} U_i^1 &= r^2(U_{i+1}^0 + U_{i-1}^0) + 2(1-r^2)U_i^0 - U_i^{-1}, \\ \frac{U_i^1 - U_i^{-1}}{2\Delta t} &= \sin(4\pi x_i) \end{aligned}$$

  Solve the last with respect to U_i^{-1} and insert this into the first expressions gives the following expression for U_i^1:

$$\begin{aligned} U_i^1 &= \frac{r^2}{2}(U_{i+1}^0 + U_{i-1}^0) + (1-r^2)U_i^0 + \Delta t \sin(4\pi x_i), \qquad i = 1, \dots, M-1 \quad (2) \\ U_i^0 &= 3\sin(\pi x_i). \end{aligned}$$

- With $\Delta x = \Delta t = 0.1$ the pararmeter $r = 1$ and thus $r^2 - 1 = 0$. To find an approximation $U_1^2 \approx u(0.1, 0.2)$ we need *in this particular case* only to calculate U_2^1 from (2) first, and then U_1^2 from (1). Use also the boundary condition $U_0^n = 0$: So

$$\begin{aligned} U_2^1 &= \frac{3}{2}(\sin(0.1\pi) + \sin(0.3\pi)) + 0.1\sin(0.8\pi) = 1.735830, \\ U_1^2 &= U_2^1 - U_1^0 = \frac{3}{2}\Big((\sin 0.3\pi) - \sin(0.1\pi)\Big) + 0.1 \cdot \sin(0.8\pi) = 0.808779. \end{aligned}$$

To compare, the exact solution is $u(0.1, 0.2) = 0.79449$.

Fourier Transform

$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega x}\, \mathrm{d}\omega$	$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x}\, \mathrm{d}x$
$f * g(x)$	$\sqrt{2\pi}\hat{f}(\omega)\hat{g}(\omega)$
$f'(x)$	$i\omega\hat{f}(\omega)$
e^{-ax^2}	$\frac{1}{\sqrt{2a}} e^{-\omega^2/4a}$
$e^{-a\lvert x\rvert}$	$\sqrt{\frac{2}{\pi}} \frac{a}{\omega^2 + a^2}$
$\frac{1}{x^2 + a^2}$	$\sqrt{\frac{\pi}{2}} \frac{e^{-a\lvert\omega\rvert}}{a}$
$f(x) = 1$ for $\lvert x\rvert < a$, 0 otherwise	$\sqrt{\frac{2}{\pi}} \frac{\sin \omega a}{\omega}$

Laplace Transform

$f(t)$	$F(s) = \int_0^{\infty} e^{-st} f(t)\, \mathrm{d}t$
$f'(t)$	$sF(s) - f(0)$
$tf(t)$	$-F'(s)$
$e^{at} f(t)$	$F(s-a)$
$\cos(\omega t)$	$\frac{s}{s^2 + \omega^2}$
$\sin(\omega t)$	$\frac{\omega}{s^2 + \omega^2}$
t^n	$\frac{n!}{s^{n+1}}$
e^{at}	$\frac{1}{s-a}$
$f(t-a)u(t-a)$	$e^{-sa} F(s)$
$\delta(t-a)$	e^{-as}
$f * g(t)$	$F(s)G(s)$

Numerics

- Newton's method: $x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$.
- Newton's method for system of equations: $\vec{x}_{k+1} = \vec{x}_k - JF(\vec{x}_k)^{-1}F(\vec{x}_k)$, with $JF = (\partial_j f_i)$.
- Lagrange interpolation: $p_n(x) = \sum_{k=0}^{n} \frac{l_k(x)}{l_k(x_k)} f_k$, with $l_k(x) = \prod_{j \neq k}(x - x_j)$.
- Interpolation error: $\epsilon_n(x) = \prod_{k=0}^{n}(x - x_k)\frac{f^{(n+1)}(t)}{(n+1)!}$.
- Chebyshev points: $x_k = \cos\left(\frac{2k+1}{2n+2}\pi\right)$, $0 \leq k \leq n$.
- Newton's divided difference: $f(x) \approx f_0 + (x - x_0)f[x_0, x_1]+$ $(x - x_0)(x - x_1)f[x_0, x_1, x_2] + \cdots + (x - x_0)(x - x_1)\cdots(x - x_{n-1})f[x_0, \ldots, x_n]$, with $f[x_0, \ldots, x_k] = \frac{f[x_1,\ldots x_k] - f[x_0,\ldots,x_{k-1}]}{x_k - x_0}$.
- Trapezoid rule: $\int_a^b f(x)\,\mathrm{d}x \approx h\left[\frac{1}{2}f(a) + f_1 + f_2 + \cdots + f_{n-1} + \frac{1}{2}f(b)\right]$.
  Error of the trapezoid rule: $|\epsilon| \leq \frac{b-a}{12}h^2 \max_{x\in[a,b]}|f''(x)|$.
- Simpson rule: $\int_a^b f(x)\,\mathrm{d}x \approx \frac{h}{3}\left[f_0 + 4f_1 + 2f_2 + 4f_3 + \cdots + 2f_{n-2} + 4f_{n-1} + f_n\right]$.
  Error of the Simpson rule: $|\epsilon| \leq \frac{b-a}{180}h^4 \max_{x\in[a,b]}|f^{(4)}(x)|$.
- Gauss–Seidel iteration: $\mathbf{x}^{(m+1)} = \mathbf{b} - \mathbf{L}\mathbf{x}^{(m+1)} - \mathbf{U}\mathbf{x}^{(m)}$, with $\mathbf{A} = \mathbf{I} + \mathbf{L} + \mathbf{U}$.
- Jacobi iteration: $\mathbf{x}^{(m+1)} = \mathbf{b} + (\mathbf{I} - \mathbf{A})\mathbf{x}^{(m)}$.
- Euler method: $\mathbf{y}_{n+1} = \mathbf{y}_n + h\mathbf{f}(x_n, \mathbf{y}_n)$.
- Improved Euler (Heun's) method: $\mathbf{y}_{n+1} = \mathbf{y}_n + \frac{1}{2}h[\mathbf{f}(x_n, \mathbf{y}_n) + \mathbf{f}(x_n + h, \mathbf{y}^*_{n+1})]$, where $\mathbf{y}^*_{n+1} = \mathbf{y}_n + h\mathbf{f}(x_n, \mathbf{y}_n)$.
- Classical Runge–Kutta method: $\mathbf{k}_1 = h\mathbf{f}(x_n, \mathbf{y}_n)$,
  $\mathbf{k}_2 = h\mathbf{f}(x_n + h/2, \mathbf{y}_n + \mathbf{k}_1/2)$, $\mathbf{k}_3 = h\mathbf{f}(x_n + h/2, \mathbf{y}_n + \mathbf{k}_2/2)$,
  $\mathbf{k}_4 = h\mathbf{f}(x_n + h, \mathbf{y}_n + \mathbf{k}_3)$, $\mathbf{y}_{n+1} = \mathbf{y}_n + \frac{1}{6}\mathbf{k}_1 + \frac{1}{3}\mathbf{k}_2 + \frac{1}{3}\mathbf{k}_3 + \frac{1}{6}\mathbf{k}_4$.
- Backward Euler method: $\mathbf{y}_{n+1} = \mathbf{y}_n + h\mathbf{f}(x_{n+1}, \mathbf{y}_{n+1})$.
- Finite differences: $\frac{\partial u}{\partial x}(x, y) \approx \frac{u(x+h,y) - u(x-h,y)}{2h}$, $\frac{\partial^2 u}{\partial x^2}(x, y) \approx \frac{u(x+h,y) - 2u(x,y) + u(x-h,y)}{h^2}$.
- Crank–Nicolson method for the heat equation: $r = \frac{k}{h^2}$,
  $(2 + 2r)u_{i,j+1} - r(u_{i+1,j+1} + u_{i-1,j+1}) = (2 - 2r)u_{ij} + r(u_{i+1,j} + u_{i-1,j})$.