LAPLACE TRANSFORM, FOURIER TRANSFORM AND DIFFERENTIAL EQUATIONS

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These notes for TMA4135 (first seven weeks) are based on Erwin Kreyszig's book [2], Dag Wessel-Berg's video: *http://video.adm.ntnu.no/serier/4fe2d4d3dbe03*, and references [1, 3, 4] (among them [1] is very short and readable).

CONTENTS

1. LAPLACE TRANSFORM

1.1. Basic facts.

Definition 1.1. *Let $f(t), t \geq 0$ be a given function. We call*

$$F(s) := \int_0^\infty e^{-st} f(t)\, dt,$$

*the **Laplace transform** of $f(t)$ and write*

$$F = \mathcal{L}(f),\ f = \mathcal{L}^{-1}F.$$

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Remark: One can prove that the Laplace transform $\mathcal{L}$ is injective (see page 9 in [1]), that is the reason why $\mathcal{L}^{-1}$ is well defined (for a precise formula of $\mathcal{L}^{-1}$, see page 10 in [1]). To compute Laplace transforms, we need:

$$d(fg) = fdg + gdf, \ \int_a^b df = f(b) - f(a), \tag{1}$$

where $df := f'(t)dt$.

Example 1:

$$\mathcal{L}(1) = \frac{1}{s}, \ \ \mathcal{L}^{-1}(\frac{1}{s}) = 1, \ \ s > 0. \tag{2}$$

Example 2:

$$\mathcal{L}(e^{kt}) = \frac{1}{s-k}, \ \ \mathcal{L}^{-1}(\frac{1}{s-k}) = e^{kt}, \ \ s > k. \tag{3}$$

Example 3: $\mathcal{L}(e^{t^2})$ does not exist,

$$\int_0^\infty e^{-st}e^{t^2}\, dt = \infty,$$

for all real number s.

Remark: Laplace transform is linear: By linearity, we mean for all real numbers a, b,

$$\mathcal{L}(af + bg) = a\mathcal{L}(f) + b\mathcal{L}(g). \tag{4}$$

Application 1:

$$\mathcal{L}(3 + 2e^{5t}) = 3\mathcal{L}(1) + 2\mathcal{L}(e^{5t}) = \frac{5(s-3)}{s-5}, \ s > 5.$$

Application 2: Since

$$\frac{1}{s^2 - 3s + 2} = \frac{1}{(s-1)(s-2)} = \frac{1}{s-2} - \frac{1}{s-1}.$$

linearity gives

$$\mathcal{L}^{-1}(\frac{1}{s^2 - 3s + 2}) = \mathcal{L}^{-1}(\frac{1}{s-2}) - \mathcal{L}^{-1}(\frac{1}{s-1}) = e^{2t} - e^t.$$

Proposition 1.2. *$\mathcal{L}(t^n) = \frac{n!}{s^{n+1}}$, for $n = 1, 2, \cdots$, and $s > 0$.*

1.2. **Laplace transform of derivatives, ODEs.**

Theorem 1.3 (Laplace transform of the derivative)**.**

$$\mathcal{L}(f') = s\mathcal{L}(f) - f(0).$$

Example: Solve:

$$y' = y, \ y(0) = 1.$$

The answer is

$$y(t) = e^t.$$

Remark: Apply the theorem to f', we get

$$\mathcal{L}(f'') = sL(f') - f'(0) = s(s\mathcal{L}(f) - f(0)) - f'(0) = s^2\mathcal{L}(f) - sf(0) - f'(0).$$

Remark: Apply the Laplace transform to a differential equation

$$y'' + ay' + by = c(t),\ a, b \in \mathbb{R},$$

then we get

$$s^2Y - sy(0) - y'(0) + a(sY - y(0)) + bY = C,$$

i.e

$$(s^2 + as + b)Y = (s + a)y(0) + y'(0) + C,$$

thus the inverse transform gives the solution

$$y = \mathcal{L}^{-1}(Y) = \mathcal{L}^{-1}\left(\frac{(s + a)y(0) + y'(0) + C}{s^2 + as + b}\right).$$

Example: Consider

$$y'' + 4y' + 4y = 0,\ \ y(0) = 0, y'(0) = 1.$$

then the above formula gives

$$y = \mathcal{L}^{-1}\left(\frac{1}{s^2 + 4s + 4}\right) = \mathcal{L}^{-1}\left(\frac{1}{(s + 2)^2}\right).$$

How to compute the inverse Laplace transform of $\frac{1}{(s+2)^2}$? Is it related to $\mathcal{L}^{-1}(\frac{1}{s^2}) = t$? We will answer them in the next section.

1.3. **More Laplace transforms.**

1.3.1. *s-Shifting.* The following formula

$$\int_0^{\infty} e^{-st}e^{at}f(t)\ dt = F(s - a),\ F(s),$$

gives

Theorem 1.4 (s-Shifting theorem)**.**

$$\mathcal{L}(e^{at}f(t)) = F(s - a).$$

Example:

$$\mathcal{L}^{-1}\left(\frac{1}{(s + 2)^2}\right) = e^{-2t}t.$$

Example:

$$\mathcal{L}(e^{kt}) = \frac{1}{s - k}.$$

Take $k = iw$,

$$\mathcal{L}(e^{iwt}) = \frac{1}{s - iw} = \frac{s + iw}{s^2 + w^2}.$$

Euler's formula (see the appendix) gives

$$\mathcal{L}(\cos wt) = \frac{s}{s^2 + w^2},\ \mathcal{L}(\sin wt) = \frac{w}{s^2 + w^2}. \tag{5}$$

Example:

$$\mathcal{L}^{-1}(\frac{s + 2}{s^2 + 4}) = \mathcal{L}^{-1}(\frac{s}{s^2 + 4}) + \mathcal{L}^{-1}(\frac{2}{s^2 + 4}) = \cos 2t + \sin 2t.$$

Exercises:

1. Find the inverse Laplace transform of $\frac{s}{s^2+2s+2}$. The answer is

$$\mathcal{L}^{-1}(\frac{s}{s^2+2s+2}) = e^{-t}(\cos t - \sin t).$$

2. Solve $y'' - y = t,\ y(0) = y'(0) = 1$. The answer is

$$y = e^t + \frac{1}{2}(e^t - e^{-t}) - t.$$

Definition 1.5 ($\sinh t$ and $\cos t$)**.**

$$\sinh t := \frac{e^t - e^{-t}}{2}, \ \ \cosh t := \frac{e^t + e^{-t}}{2}.$$

Exercises:

3. Compute Laplace transform of $\sinh t$ and $\cosh t$. The answer is

$$\mathcal{L}(\sinh t) = \frac{1}{s^2-1},\ \mathcal{L}(\cosh t) = \frac{s}{s^2-1}.$$

4. Compute Laplace transform of

$$f(t) = 1 \text{ if } 3 < t < 4; \quad f(t) = 0 \text{ otherwise.}$$

The answer is

$$\mathcal{L}(f) = \frac{e^{-3s} - e^{-4s}}{s}.$$

1.3.2. *Laplace transform of integrals.* Put

$$g(t) = \int_0^t f(\tau)\, d\tau,$$

then

$$g' = f,\ g(0) = 0.$$

Thus

$$F = \mathcal{L}(f) = \mathcal{L}(g') = sG - g(0) = sG$$

gives $G = \frac{F}{s}$, i.e.

$$\mathcal{L}\left(\int_0^t f(\tau)\, d\tau\right) = \frac{\mathcal{L}(f)}{s}.$$

Exercises:

1. Show that

$$\mathcal{L}^{-1}\left(\frac{1}{s(s^2+1)}\right) = 1 - \cos t.$$

2. Show that

$$\mathcal{L}^{-1}\left(\frac{1}{s} \cdot \frac{1}{s(s-1)}\right) = e^t - 1 - t.$$

1.3.3. *Using step functions.*

Definition 1.6 (Step function)**.** *Let $a \geq 0$, the step function $u(t-a)$ is defined as follows*

$$u(t-a) = 0, \text{ for } 0 \leq t < a; \quad u(t-a) = 1, \text{ for } t \geq a.$$

In case $a = 0$ we call $u(t)$ the Heaviside function.

Exercise:

1. Draw the graphs of

$$f(t) = u(t-1) - u(t-3),$$

$$f(t) = (u(t) - u(t-\pi)) \sin t,$$

$$f(t) = u(t) + u(t-1) + \cdots + u(t-n) + \cdots .$$

2. Show that

$$\mathcal{L}(u(t-a)) = \frac{e^{-as}}{s}.$$

3. Compare the graphs of $u(t-a)f(t-a)$ with that of $f(t)$.

Theorem 1.7 (t-Shifting theorem)**.**

$$\mathcal{L}(u(t-a)f(t-a)) = e^{-as}\mathcal{L}(f).$$

Example: Since

$$\frac{1}{s-2} = \mathcal{L}(e^{2t}),$$

we have

$$\mathcal{L}^{-1}(e^{-s}\frac{1}{s-2}) = u(t-1)e^{2(t-1)}.$$

Exercise:

1. Compute Laplace transform of

$$f(t) = \sum_{n=0}^{\infty} u(t-n).$$

(Hint: $\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$ for $|r| < 1$) Answer

$$\mathcal{L}\left(\sum_{n=0}^{\infty} u(t-n)\right) = \frac{1}{s(1-e^{-s})}.$$

RC-Circuit equation (see page 29 section 1.5 and page 93 section 2.9 of Kreyszig's book): R, C postive constants, $i(t), e(t)$ functions:

$$Ri(t) + \frac{1}{C}\int_0^t i(\tau)\, d\tau = e(t).$$

Apply the Laplace transform, we get

$$RI(s) + \frac{1}{C} \cdot \frac{I(s)}{s} = E(s),$$

i.e.

$$I(s) = \frac{E(s)}{R + \frac{1}{Cs}} = \frac{s}{s + \frac{1}{RC}} \frac{E(s)}{R}.$$

Exercise:

2. Compute $i(t)$ when

$$e(t) = u(t-1) - u(t-2), \quad R = C = 1.$$

Answer

$$i(s) = u(t-1)e^{-(t-1)} - u(t-2)e^{-(t-2)}.$$

1.3.4. *Dirac delta function.* The delta function $\delta(t-a)$ is defined by

$$\int_0^\infty f(t)\delta(t-a)\,dt = f(a).$$

In case $f(t) = e^{-st}$, we have

$$\int_0^\infty e^{-st}\delta(t-a)\,dt = e^{-as}.$$

Thus

$$\mathcal{L}(\delta(t-a)) = e^{-as}.$$

Exercise: solve

$$y'' + y = \delta(t-1), \quad y(0) = y'(0) = 0.$$

Answer

$$y = \mathcal{L}^{-1}(e^{-s}\mathcal{L}(\sin t)) = u(t-1)\sin(t-1).$$

1.3.5. *Convolution.* Let $f(t), g(t)$ be two functions for $t \geq 0$.

Definition 1.8 (Convolution of f and g)**.**

$$(f \star g)(t) := \int_0^t f(\tau)g(t-\tau)\,d\tau, \qquad t \geq 0.$$

Examples:

$$1 \star t = \frac{t^2}{2},$$

$$e^t \star e^t = te^t,$$

$$f(t) \star 1 = \int_0^t f(\tau)\,d\tau.$$

Theorem 1.9 (Laplace transform of convolution)**.**

$$\mathcal{L}(f \star g) = \mathcal{L}(f) \cdot \mathcal{L}(g).$$

Exercises:

1. Compute $t^m \star t^n$. Hint: use $t^m \star t^n = \mathcal{L}^{-1}\mathcal{L}(t^m \star t^n)$, answer

$$t^m \star t^n = \frac{m!n!}{(m+n+1)!} t^{m+n+1},\ m, n = 0, 1, \cdots$$

2. Compute $\mathcal{L}^{-1}(\frac{1}{(s^2+1)^2})$. Hint: use $\mathcal{L}^{-1}(\frac{1}{(s^2+1)^2}) = \mathcal{L}^{-1}(\mathcal{L}(\sin t) \cdot \mathcal{L}(\sin t))$. Answer

$$\mathcal{L}^{-1}(\frac{1}{(s^2+1)^2}) = \frac{\sin t - t\cos t}{2}.$$

3. Solve $y'' + y = \sin t,\ y(0) = 0,\ y'(0) = 1$. Hint: use Exercise 2. Answer

$$y = \sin t + \frac{\sin t - t\cos t}{2} = \frac{3\sin t - t\cos t}{2}.$$

4. Solve: $y - \int_0^t (t-\tau)y(\tau)\,d\tau = 1$. Hint: use $\int_0^t (t-\tau)y(\tau)\,d\tau = y \star t$. Answer

$$y = \frac{e^t + e^{-t}}{2} = \cosh t.$$

1.3.6. *Non-homogeneous linear ODEs.* Consider

$$y'' + by' + cy = r(t),$$

given $y(0)$ and $y'(0)$, we have

$$s^2 Y - sy(0) - y'(0) + b(sY - y(0)) + cY = R(s).$$

Thus

$$Y = \frac{1}{s^2+bs+c} \cdot R(s) + \frac{sy(0) + y'(0) + by(0)}{s^2+bs+c} := K(s) \cdot R(s) + G(s),$$

we get

$$y = k \star r + g.$$

Example: Consider

$$y'' + y = r(t),\ \ y(0) = y'(0) = 0.$$

Apply the Laplcae transform, we have

$$s^2 Y + Y = \mathcal{L}(r).$$

Thus

$$Y = \frac{1}{s^2+1} \cdot \mathcal{L}(r),$$

which gives

$$y(t) = \sin t \star r.$$

1.3.7. *Derivative of the Laplace transform.* Apply differential to

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt,$$

we get

$$F'(s) = \int_0^\infty \frac{d(e^{-st})}{ds} f(t)\, dt = \int_0^\infty e^{-st} \cdot (-tf(t))\, dt = \mathcal{L}(-tf(t)).$$

Example 1:

$$\mathcal{L}(t \sin t) = -(\frac{1}{s^2+1})' = \frac{2s}{(s^2+1)^2}.$$

Example 2: Let $F(s) = \ln(1 + s^{-2})$. Then

$$F' = (\ln(1+s^2) - \ln(s^2))' = \frac{2s}{1+s^2} - \frac{2}{s}.$$

Thus

$$\mathcal{L}^{-1}(F') = 2\cos t - 2.$$

By the above theorem, we have

$$\mathcal{L}^{-1}(F') = -tf(t).$$

thus

$$f(t) = \mathcal{L}^{-1}(\ln(1+s^{-2})) = \frac{2 - 2\cos t}{t}.$$

1.3.8. *System of differential equations.* Look at this example:

$$y_1' = -y_1 + y_2; \;\; y_2' = -y_1 - y_2 + f(t), \;\; y_1(0) = y_2(0) = 0.$$

Apply the Laplace transform, we get

$$sY_1 = -Y_1 + Y_2; \;\; sY_2 = -Y_1 - Y_2 + F(s).$$

Thus

$$(s+1)Y_1 - Y_2 = 0;$$

$$Y_1 + (s+1)Y_2 = F(s).$$

The first equation gives $Y_2 = (s+1)Y_1$, together with the second, we have

$$Y_1 = F(s)(1 + (s+1)^2)^{-1}.$$

Use the first equation again,

$$Y_2 = F(s)(s+1)(1 + (s+1)^2)^{-1}.$$

Thus

$$y_1 = f(t) \star (e^{-t} \sin t), \; y_2 = f(t) \star (e^{-t} \cos t).$$

when $f(t) = e^{-t}$, we get

$$y_1 = \int_0^t e^{-(t-\tau)} e^{-\tau} \sin \tau \, d\tau = e^{-t}(1 - \cos t), \;\; y_2 = e^{-t} \sin t.$$

1.3.9. *Homework for Laplace transform.* Please compute Laplace transform of

1. $f(t) = t,$ if $0 \le t \le a,\ \ f(t) = 0,$ if $t > a$. Answer

$$F(s) = \frac{1}{s^2} - \frac{e^{-as}}{s^2} - a\frac{e^{-as}}{s}.$$

2. $f(t) = u(t-\pi)\sin t$. Answer $F = -\frac{e^{-\pi s}}{s^2+1}$.
3. Solve the following equation

$$i'(t) + 2\,i(t) + \int_0^t i(\tau)\,d\tau = \delta(t-1),\ i(0) = 0.$$

Answer

$$i(t) = u(t-1)(e^{-(t-1)} - e^{-(t-1)}(t-1)).$$

2. Fourier analysis

2.1. **Complex and real Fourier series.**

2.1.1. *Complex Fourier series.* Fix $p > 0$, if

$$f(x+p) = f(x),\ \forall\, x \in \mathbb{R},$$

then call f a *periodic function with period* p.

Example: periodic function:

1. A polynomial is periodic if and only if it is a constant;
2. $e^{\lambda x}$ has period 2π if and only if $\lambda = in,\ \ n \in \mathbb{Z}$.

The main theorem in Fourier analysis is the following:

Theorem 2.1 (Fourier 1807)**.** *If f has period 2π and is smooth enough then we have*

$$f(x) = \sum_{n\in\mathbb{Z}} c_n\, e^{inx},\ \forall\, x \in \mathbb{R}.$$

——*The proof (see Page 63 in* [3]*) is not assumed in this course.*

What does "smooth enough" mean? It means that f is piecewise smooth and

$$f(x_0) = \frac{f(x_0+) + f(x_0-)}{2},$$

if f is not smooth at x_0.

How to compute c_n ? We shall prove that

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx.$$

In fact , by the above theorem, we have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx = \sum_{m\in\mathbb{Z}} \frac{c_m}{2\pi}\int_{-\pi}^{\pi} e^{imx}e^{-inx}\,dx.$$

If $m = n$ then

$$\int_{-\pi}^{\pi} e^{imx}e^{-inx}\,dx = \int_{-\pi}^{\pi} 1\,dx = 2\pi.$$

If $m \neq n$ then

$$\int_{-\pi}^{\pi} e^{i(m-n)x}\, dx = \int_{-\pi}^{\pi} d(\frac{e^{i(m-n)x}}{i(m-n)}) = 0.$$

Thus

$$\sum_{m\in\mathbb{Z}} \frac{c_m}{2\pi} \int_{-\pi}^{\pi} e^{imx} e^{-inx}\, dx = c_n.$$

Definition 2.2. *We call*

$$f(x) = \sum_{n\in\mathbb{Z}} c_n e^{inx}$$

*the **complex Fourier series** of f and*

$$c_n := \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx}\, dx,\ n \in \mathbb{Z},$$

*the **complex Fourier coefficients** of f.*

Example: Consider

$$f(x) = 1,\ 0 < x < \pi;\ \ f(x) = -1,\ -\pi < x < 0,$$

and

$$f(0) = f(\pi) = f(-\pi) = 0.$$

Then we know f is smooth enough and

$$2\pi c_n = \int_{-\pi}^{\pi} f(x) e^{-inx}\, dx = \int_{0}^{\pi} e^{-inx}\, dx - \int_{-\pi}^{0} e^{-inx}\, dx.$$

Since

$$\int_{0}^{\pi} e^{-inx}\, dx = \int_{0}^{\pi} d(\frac{e^{-inx}}{-in}) = \frac{(-1)^n - 1}{-in},$$

and

$$\int_{-\pi}^{0} e^{-inx}\, dx = \int_{-\pi}^{0} d(\frac{e^{-inx}}{-in}) = \frac{1 - (-1)^n}{-in},$$

we have

$$2\pi c_n = \frac{2(1 - (-1)^n)}{in},$$

i.e.

$$c_n = \frac{2}{in\pi},\ n\ odd;\ \ c_n = 0,\ n\ even.$$

Thus the complex fourier series of f is

$$f(x) = \sum_{m\in\mathbb{Z}} \frac{2}{i(2m+1)\pi} e^{i(2m+1)x}.$$

2.1.2. *(Real) Fourier series.* In the previous example, we have

$$f(x) = \frac{2}{i\pi}\left(e^{ix} + \frac{e^{3ix}}{3} + \cdots\right) + \frac{2}{i\pi}\left(\frac{e^{-ix}}{-1} + \frac{e^{-3ix}}{-3} + \cdots\right),$$

thus

$$f(x) = \frac{2}{i\pi}\left((e^{ix} - e^{-ix}) + \frac{e^{3ix} - e^{-3ix}}{3} + \cdots\right).$$

Euler's formula gives

$$e^{inx} - e^{-inx} = 2i \sin nx,$$

thus

$$f(x) = \frac{4}{\pi}\left(\sin x + \frac{\sin 3x}{3} + \cdots\right),$$

In particular, it gives

$$1 = f(\frac{\pi}{2}) = \frac{4}{\pi}\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\right).$$

Thus

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4},$$

which is a famous formula obtained by Leibniz in 1673 from geometric considerations.

For a general function f, by the Euler formula, we have

$$f(x) = \sum c_n e^{inx} = \sum c_n(\cos nx + i \sin nx),$$

which gives

$$f(x) = c_0 + \sum_{n=1}^{\infty} c_n(\cos nx + i \sin nx) + \sum_{n=1}^{\infty} c_{-n}(\cos nx - i \sin nx).$$

Thus we have

$$f(x) = c_0 + \sum_{n=1}^{\infty} \left((c_n + c_{-n}) \cos nx + i(c_n - c_{-n}) \sin nx\right),$$

Recall that

$$(c_n + c_{-n}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)(e^{-inx} + e^{inx})\, dx = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx\, dx,$$

and

$$i(c_n - c_{-n}) = \frac{i}{2\pi}\int_{-\pi}^{\pi} f(x)(e^{-inx} - e^{inx})\, dx = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx\, dx,$$

thus we get

Theorem 2.3. *If f has period 2π and is smooth enough then it has the following* ***Fourier series*** *expansion*

$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx),$$

where a_0, a_n, b_n are the ***Fourier coefficients*** *of f such that*

$$a_0 = c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, dx,$$

and for $n = 1, 2\cdots$, *we have*

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx,$$

and

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx.$$

Example: Consider

$$f(x) = 0, \ \ -\pi < x < 0; \quad f(x) = x, \ \ 0 \leq x < \pi.$$

Then

$$2\pi a_0 = \int_{-\pi}^{\pi} f(x)\, dx = \int_0^{\pi} x\, dx = \frac{\pi^2}{2},$$

and

$$\pi a_n = \int_{-\pi}^{\pi} f(x)\cos nx\, dx = \int_0^{\pi} x\cos nx\, dx = \int_0^{\pi} x\, d(\frac{\sin nx}{n}) = -\int_0^{\pi}\frac{\sin nx}{n}\, dx.$$

Since

$$-\int_0^{\pi}\frac{\sin nx}{n}\, dx = \int_0^{\pi} d(\frac{\cos nx}{n^2}) = \frac{(-1)^n - 1}{n^2},$$

we get

$$a_0 = \frac{\pi}{4}, \ \ a_{2m} = 0, \ a_{2m-1} = \frac{-2}{(2m-1)^2\pi}, \ \ m = 1, 2\cdots.$$

Moreover, we have

$$\pi b_n = \int_{-\pi}^{\pi} f(x)\sin nx\, dx = \int_0^{\pi} x\sin nx\, dx = \int_0^{\pi} x\, d(\frac{-\cos nx}{n}) = \frac{\pi(-1)^{n+1}}{n} + \int_0^{\pi}\frac{\cos nx}{n}\, dx,$$

Notice that

$$\int_0^{\pi}\frac{\cos nx}{n}\, dx = \int_0^{\pi} d(\frac{\sin nx}{n^2}) = 0,$$

thus

$$b_n = \frac{(-1)^{n+1}}{n}.$$

Thus

$$f(x) = \frac{\pi}{4} - \frac{2}{\pi}\left(\cos x + \frac{\cos 3x}{3^2} + \cdots\right) + \left(\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \cdots\right).$$

Take $x = 0$ then we get

$$0 = \frac{\pi}{4} - \frac{2}{\pi}(1 + \frac{1}{3^2} + \cdots),$$

i.e.

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}.$$

Exercise: Use $1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$ to prove that

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}.$$

Proposition 2.4. *Put $\delta_{mn} = 1$ if $m = n$ and $\delta_{mn} = 0$ if $m \neq n$ then*

$$\int_{-\pi}^{\pi} \cos nx \cos mx \, dx = \int_{-\pi}^{\pi} \sin nx \sin mx \, dx = \pi \delta_{mn}, \quad m, n = 1, 2, \cdots,$$

and

$$\int_{-\pi}^{\pi} \cos nx \, dx = 2\pi \delta_{n0}, \quad \int_{-\pi}^{\pi} \cos nx \sin mx = 0, \quad m, n = 0, 1, 2, \cdots.$$

Proof. Follows from the Euler formula

$$\cos nx = \frac{e^{inx} + e^{-inx}}{2}, \ \sin nx = \frac{e^{inx} - e^{-inx}}{2i},$$

and

$$\int_{-\pi}^{\pi} e^{inx} e^{-imx} \, dx = 2\pi \delta_{mn}.$$

Try to give the details yourself. □

Exercise: Try to use the above proposition to prove the formulas for a_n, b_n in Theorem 2.3.

2.2. **Fourier Sine and Cosine series.**

Definition 2.5. *We say that f is odd if $f(-x) = -f(x)$; f is even if $f(-x) = f(x)$.*

Example: For every positive integer n, we know that $\cos nx$ is even and $\sin nx$ is odd.

Application: If f is even then

$$\int_{-\pi}^{\pi} f(x) \, dx = 2 \int_{0}^{\pi} f(x) \, dx.$$

If f is odd then

$$\int_{-\pi}^{\pi} f(x) \, dx = 0.$$

In particular, if f is odd then all

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx = 0;$$

if f is even then all

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx = 0.$$

Thus we get:

Theorem 2.6. *Assume that f has period 2π and is smooth enough. If f is **odd** then it can be written as a **Fourier sine series***

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx, \quad b_n = \frac{2}{\pi} \int_{0}^{\pi} f(x) \sin nx \, dx.$$

*If f is **even** then it can be written as a **Fourier cosine series***

$$f(x) = \frac{1}{\pi} \int_{0}^{\pi} f(x) \, dx + \sum_{n=1}^{\infty} a_n \cos nx, \ a_n = \frac{2}{\pi} \int_{0}^{\pi} f(x) \cos nx \, dx.$$

Odd or Even extension: Let f be a function in $(0,\pi)$. Then we can extend f to an odd function, say f_o such that

$$f_o(-x) = -f(x),\ x \in (0,\pi);$$

we can also extend f to an even function, say f_e such that

$$f_e(-x) = f(x),\ x \in (0,\pi).$$

Exercise:

1. Draw the graph of the odd extension f_o and the even extension f_e of the following function

$$f(x) = x,\ \ 0 < x < \frac{\pi}{2};\ \ f(x) = \frac{\pi}{2},\ \ \frac{\pi}{2} < x < \pi.$$

2. Find the Fourier cosine series of f_e. Answer

$$f_e(x) = \frac{3\pi}{8} + \frac{2}{\pi}\left(-\cos x - \frac{2\cos 2x}{2^2} - \frac{\cos 3x}{3^2} - \frac{\cos 5x}{5^2} - \cdots\right).$$

3. Find the Fourier sine series of f_o. Answer

$$\begin{aligned} f_o(x) \ &= \ \left(\frac{2}{\pi}+1\right)\sin x + \left(0-\frac{1}{2}\right)\sin 2x + \left(\frac{-2}{3^2\pi}+\frac{1}{3}\right)\sin 3x \\ &\quad + \left(0-\frac{1}{4}\right)\sin 4x + \left(\frac{2}{5^2\pi}+\frac{1}{5}\right)\sin 5x + \cdots . \end{aligned}$$

2.3. **More Fourier series.**

2.3.1. *Parseval's identity.* Let f be a smooth enough function with period 2π. Consider the complex Fourier series expansion of f

$$f = \sum_{n\in\mathbb{Z}} c_n e^{inx}.$$

We have

$$\int_{-\pi}^{\pi} |f(x)|^2\,dx = \int_{-\pi}^{\pi} \sum_{n,m\in\mathbb{Z}} c_n\overline{c_m} e^{inx-imx}\,dx.$$

Now use that

$$\int_{-\pi}^{\pi} e^{inx-imx}\,dx = 0,$$

if $m \neq n$ and

$$\int_{-\pi}^{\pi} e^{inx-imx}\,dx = 2\pi,$$

if $m = n$. We get the following *Parseval identity*

$$\int_{-\pi}^{\pi} |f(x)|^2\,dx = 2\pi \sum_{n\in\mathbb{Z}} |c_n|^2.$$

Example: Consider the last example in section 2.1.1:

$$f(x) = 1,\ 0 < x < \pi;\ \ f(x) = -1,\ -\pi < x < 0,$$

and

$$f(0) = f(\pi) = f(-\pi) = 0.$$

We know that f has the following complex Fourier series expansion:

$$f(x) = \sum_{m\in\mathbb{Z}} \frac{2}{i(2m+1)\pi} e^{i(2m+1)x}.$$

Thus the Parseval identity gives

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2\,dx = 1 = \sum_{m\in\mathbb{Z}} \frac{4}{(2m+1)^2\pi^2} = \frac{8}{\pi^2}\left(1+\frac{1}{3^2}+\frac{1}{5^2}+\cdots\right),$$

which gives another proof of

$$1+\frac{1}{3^2}+\frac{1}{5^2}+\cdots = \frac{\pi^2}{8}.$$

2.3.2. *Resonance phenomenon.* Let us consider

$$(\star): \quad y''(t)+\omega^2 y(t) = \sin\alpha t,$$

where $\alpha \neq \omega$ are positive constants. We call

$$y(t) = A\cos\alpha t + B\sin\alpha t,$$

a *steady-state solution* of $(\star)$. A direct computation gives

$$A = 0, \quad B = \frac{1}{\omega^2-\alpha^2}.$$

The "resonance phenomenon" can be explained by the following fact: the amplitude $|B|$ of the steady-state solution is very big if α is very close to ω (the frequency of the system). In general, a steady solution of

$$y''(t)+\omega^2 y(t) = a_n\cos nt + b_n \sin nt,$$

is defined as a solution of the following form

$$y_n(t) := A_n\cos nt + B_n\sin nt.$$

Moreover, consider the following equation

$$(\star\star) \quad y''(t)+\omega^2 y(t) = \gamma(t),$$

Assume that

$$\gamma(t) = \sum_{n=0}^{\infty} a_n\cos nt + b_n\sin nt,$$

then we call

$$y(t) := y_0+y_1+\cdots+y_n+\cdots,$$

a steay state solution of $(\star\star)$. When $\omega = N$ is a natural number, avoid resonance means that the input $\gamma(t)$ satisfies $a_N = b_N = 0$.

2.3.3. *Best L^2–approximation by $\sum_{|n|\leq N} c_n e^{inx}$.* Let f be a smooth enough function with period 2π. We hope to find an N-series,

$$F_N := \sum_{|n|\leq N} C_n e^{inx},$$

such that

$$\int_{-\pi}^{\pi} |F_N - f|^2 \, dx$$

is minimal. The main idea is to use *orthogonal decomposition*.

Orthogonal Decomposition in vector space: Let S be a subspace of a vector space V (think of $S = \mathbb{R}^2 \times \{0\}$ and V is $\mathbb{R}^3$), then we can write a vector, say v, in V as

$$v = v_S + v_{S^\perp},$$

where v_S lies in S and $v_{S^\perp}$ is orthoginal to S.

Definition 2.7. *We call v_S the orthogonal projection of v to S.*

Then it is very clear from the picture that v_S is the unique solution of the following extremal problem:

$$||v_S - v|| = \min\{||u - v|| : u \in S\}.$$

For a real proof it is enough to use

$$||u - v||^2 = ||u - v_S||^2 + ||v_{S^\perp}||^2,$$

which implies that $u = v_S$ is the unique solution.

Orthogonal Decomposition in the space of functions: In our case, we consider V as the space of *complex* functions on $[-\pi, \pi]$ with the following inner product structure:

$$(f, g) := \int_{-\pi}^{\pi} f(x)\overline{g(x)} \, dx, \quad ||f||^2 := (f, f).$$

Now S is the subspace of V spanned by e^{inx}, $|n| \leq N$. Let f be a smooth enough function with period 2π. Then we know that

$$||f_S - f|| = \min\{||u - f|| : u \in S\}$$

Thus $F_N = f_S$ (recall that f_S means the orthogonal projection of f to S) solves our extremal problem.

What is f_S ? The simplest way is to use the complex Fourier series expansion

$$f(x) = \sum_{n\in\mathbb{Z}} c_n e^{inx}.$$

Put

$$f_N = \sum_{|n|\leq N} c_n e^{inx},$$

since $\{e^{inx}\}_{n\in\mathbb{Z}}$ is an orthogonal basis of V we have

$$(f_N, f - f_N) = 0,$$

which implies that f_N is the orthogonal projection of f to S. Now we have

$$f_N = f_S,$$

which gives:

Theorem 2.8. *Complex Fourier series expansion solves the best L^2–approximation by $\sum_{|n|\leq N} c_n e^{inx}$ problem problem.*

Bessel's inequality and Parseval's identity: Notice that (try!)

$$||f||^2 = ||f_N||^2 + ||f - f_N||^2,$$

Thus we get the Bessel inequality

$$||f||^2 \geq ||f_N||^2,$$

i.e.

$$\int_{-\pi}^{\pi} |f(x)|^2 \, dx \geq 2\pi \cdot \sum_{|n|\leq N} |c_n|^2,$$

and the Parseval identity

$$\int_{-\pi}^{\pi} |f(x)|^2 \, dx = 2\pi \cdot \sum_{n\in\mathbb{Z}} |c_n|^2.$$

2.4. **Fourier transform.**

Definition 2.9. *We call*

$$\hat{f}(w) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-iwx} \, dx,$$

*the **Fourier transform** of f and write $\hat{f} = \mathcal{F}(f)$.*

Example: F1: Fourier transform of $f(x) = 1$ if $|x| < 1$ and $f(x) = 0$ otherwise:

$$\hat{f}(w) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-iwx} \, dx = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} e^{-iwx} \, dx$$

if $w \neq 0$ then

$$\int_{-1}^{1} e^{-iwx} \, dx = \int_{-1}^{1} d(\frac{e^{-iwx}}{-iw}) = \frac{e^{-iw}}{-iw} - \frac{e^{iw}}{-iw} = \frac{2\sin w}{w}.$$

Notice that

$$\lim_{w\to 0} \frac{2\sin w}{w} = 2 = \int_{-1}^{1} dx = \hat{f}(0).$$

Thus we can write

$$\hat{f}(w) = \frac{1}{\sqrt{2\pi}} \left(\frac{2\sin w}{w} \right) = \sqrt{\frac{2}{\pi}} \frac{\sin w}{w}.$$

Example: F2: Fourier transform of $f(x) = e^{-x}$ if $x > 0$ and $f(x) = 0$ otherwise:

$$\hat{f}(w) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-iwx} \, dx = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-x} e^{-iwx} \, dx = \frac{1}{\sqrt{2\pi}} \mathcal{L}(e^{-iwt})(1).$$

Recall that

$$L(e^{-iwt})(s) = \frac{1}{s + iw},$$

thus

$$\hat{f}(w) = \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{1 + iw}.$$

2.4.1. *From Complex Fourier series to inverse Fourier transform.* : Assume that f is smooth enough in $-N < x < N$ and $f = 0$ when $|x| > N$. For each $L > N$, let us define a periodic function f_L such that

$$f_L(x) = f(x), \ \ |x| < L; \ \ f_L(x + 2L) = f_L(x).$$

Then we know that

$$g_L(x) = f_L\left(\frac{Lx}{\pi}\right),$$

has period 2π and is smooth enough. Thus

$$g_L(x) = \sum c_n e^{inx}, \ \ c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} g_L(x)e^{-inx}\,dx.$$

Thus

$$f_L(x) = g_L\left(\frac{\pi x}{L}\right) = \sum c_n e^{in\frac{\pi x}{L}}.$$

Consider $v = \frac{Lx}{\pi}$, we can write

$$c_n = \frac{1}{2\pi}\int_{-L}^{L} f_L(v)e^{-in\frac{\pi v}{L}}\,d(\frac{\pi v}{L}) = \frac{1}{2L}\int_{-\infty}^{\infty} f(v)e^{-in\frac{\pi v}{L}}\,dv = \frac{\sqrt{2\pi}}{2L}\hat{f}(\frac{n\pi}{L}).$$

which gives

$$f(x) = \sqrt{\frac{\pi}{2}}\sum_{n\in\mathbb{Z}}\frac{\hat{f}(\frac{n\pi}{L})\cdot e^{in\frac{\pi x}{L}}}{L}.$$

Put

$$\Delta w = \frac{\pi}{L},$$

then we have

$$f(x) = \frac{1}{\sqrt{2\pi}}\sum_{n\in\mathbb{Z}}\hat{f}(n\Delta w)\cdot e^{ix\cdot n\Delta w}\Delta w.$$

Assume that $\hat{f}(w)e^{ixw}$ is integrable in $-\infty < x < \infty$. Let L goes to infty, the above formula gives the following *Fourier inversion formula*:

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{f}(w)e^{ixw}\,dw.$$

We say that $f(x)$ is the *inverse Fourier transform* of $\hat{f}(w)$ and write $f = \mathcal{F}^{-1}(\hat{f})$.

2.5. **Fourier inversion formula.** When do we have the Fourier inversion formula ? It is known that (see Page 141 Theorem 1.9 in [3]) the Fourier inversion formula is true if f is *smooth and rapidly decreaing*, in the sense that

$$\sup_{x\in\mathbb{R}}|x|^k|f^{(l)}(x)| < \infty, \ \ \text{for every } k, l \geq 0,$$

where $f^{(l)}$ denotes the l-th derivative of f.

Example: Let $f(x) = e^{-\frac{x^2}{2}}$. We shall use Fourier inversion formula to prove

$$\hat{f}(w) := \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}e^{-iwx}dx = e^{-\frac{w^2}{2}} = f(w). \tag{6}$$

Step 1: Look at the derivative of $\hat{f}(w)$:

$$(\hat{f})'(w) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}(-ix)e^{-iwx}dx = \mathcal{F}(-ixf(x)).$$

Notice that $(e^{-\frac{x^2}{2}})' = e^{-\frac{x^2}{2}}(-x)$, thus

$$\hat{f}'(w) = \frac{i}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-iwx} d(e^{-\frac{x^2}{2}}) = \frac{-i}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} d(e^{-iwx}) = -w\hat{f}(w).$$

Now we have

$$\left(\hat{f}(w)e^{\frac{w^2}{2}}\right)' = (-w+w)\left(\hat{f}(w)e^{\frac{w^2}{2}}\right) = 0.$$

Thus $\hat{f}(w)e^{\frac{w^2}{2}}$ is a constant, i.e.

$$\hat{f}(w)e^{\frac{w^2}{2}} \equiv \hat{f}(0)e^0 = \hat{f}(0).$$

Now we have

$$\hat{f}(w) = \hat{f}(0)e^{\frac{-w^2}{2}} = \hat{f}(0)f(w)$$

Step 2: The Fourier inversion formula implies (notce that f is smooth and rapidly decreasing)

$$f(x) = \mathcal{F}^{-1}(\hat{f}) = \hat{f}(0)\mathcal{F}^{-1}(f) = \hat{f}(0)\hat{f}(-x) = (\hat{f}(0))^2 f(x).$$

Thus $\hat{f}(0) = \pm 1$. Since

$$\hat{f}(0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}\, dx > 0,$$

we get

$$\hat{f}(0) = 1, \ \ \hat{f} = f.$$

2.5.1. *Normal distribution.* $\hat{f}(0) = 1$ gives

(7)
$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}\, dx = 1,$$

(one may also use integration on $\mathbb{R}^2$ to compute the following integral directly, see page 138 formula (6) in [3]), consider

$$u = \sqrt{t}x + \mu, \ \ t > 0, \ \ \mu \in \mathbb{R},$$

then (7) becomes the following classical formula in Gauss's normal distribution theory

(8)
$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi t}}\, e^{-\frac{(u-\mu)^2}{2t}}\, du = 1,$$

where

(9)
$$f(u \,|\, \mu, t) := \frac{1}{\sqrt{2\pi t}}\, e^{-\frac{(u-\mu)^2}{2t}},$$

is the *probability density of the normal distribution with expectation* μ *and variance* t.

2.5.2. *Inverse Laplace transform.* Recall the definition of the Laplace transform

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt$$

Extend f to a function on $\mathbb{R}$ such that

$$f(x) = 0, \quad \forall\, x < 0.$$

Thus we have

$$F(s) = \int_{-\infty}^\infty e^{-st} f(t)\, dt,$$

which gives

$$F(iw) = \sqrt{2\pi}\hat{f}(w).$$

Thus the Fourier inversion formula gives

$$\text{Laplace inversion formula :} \quad f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty \hat{f}(w) e^{itw}\, dw = \frac{1}{2\pi} \int_{-\infty}^\infty F(iw) e^{itw}\, dw.$$

2.6. **More Fourier transforms.**

2.6.1. *Derivative and convolution formulas in Fourier transform.* In this section, we only consider functions that are smooth and rapidly decreasing. The main result is the following:

Theorem 2.10. *Let $\mathcal{F}(f)(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty f(x) e^{-iwx} dx$ be the Fourier transform of f. Then*

1) $\mathcal{F}(af + bg) = a\mathcal{F}(f) + b\mathcal{F}(g)$*;*
2) $\mathcal{F}(f') = iw\mathcal{F}(f)$*;*
3) $(\mathcal{F}(f))' = -i\mathcal{F}(xf(x))$.

Proof. We only prove 2). Notice that

$$\int_{-\infty}^\infty f'(x) e^{-iwx}\, dx = \int_{-\infty}^\infty e^{-iwx}\, df = \int_{-\infty}^\infty d(e^{-iwx} f) - \int_{-\infty}^\infty f(x)\, d(e^{-iwx}).$$

Since f is smooth and repadly decreasing, we have

$$\int_{-\infty}^\infty d(e^{-iwx} f) = \lim_{x\to\infty} e^{-iwx} f(x) - \lim_{x\to-\infty} e^{-iwx} f(x) = 0.$$

Thus

$$\int_{-\infty}^\infty f'(x) e^{-iwx}\, dx = -\int_{-\infty}^\infty f(x)\, d(e^{-iwx}) = iw \int_{-\infty}^\infty f(x) e^{-iwx}\, dx,$$

which implies 2). □

Example: $\mathcal{F}(xe^{-\frac{x^2}{2}}) = -iwe^{-\frac{w^2}{2}}$: By 3), we have

$$\mathcal{F}(xe^{-\frac{x^2}{2}}) = i(\mathcal{F}(e^{-\frac{x^2}{2}}))' = i(e^{-\frac{w^2}{2}})' = -iwe^{-\frac{w^2}{2}}.$$

Convolution: Let us first recall the definition of convolution for functions f, g defined on $[0, \infty)$:

$$(f \star g)(t) := \int_0^t f(\tau) g(t-\tau)\, d\tau.$$

Notice that if we extend f, g to functions on $\mathbb{R}$ such that

$$f = g = 0, \text{ when } x \le 0.$$

Then we can write

$$(f \star g)(x) := \int_{-\infty}^{\infty} f(u)g(x-u)\,du.$$

Definition 2.11. *The convolution of two functions on $\mathbb{R}$ is defined by*

$$(f \star g)(x) := \int_{-\infty}^{\infty} f(u)g(x-u)\,du.$$

Similar as the Laplace transform, we have

$$\mathcal{F}(f \star g) = \sqrt{2\pi}\,\mathcal{F}(f) \cdot \mathcal{F}(g). \tag{10}$$

Proof. [Need integration on $\mathbb{R}^2$]. We have

$$\sqrt{2\pi}\mathcal{F}(f \star g) = \int_{-\infty}^{\infty} (f \star g)(x) e^{-iwx}\,dx = \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(u)g(x-u)\,du \right) e^{-iwx}\,dx.$$

Change the order of integration, we get

$$\sqrt{2\pi}\mathcal{F}(f \star g) = \int_{-\infty}^{\infty} f(u) \left(\int_{-\infty}^{\infty} g(x-u) e^{-iwx}\,dx \right) du = \sqrt{2\pi} \int_{-\infty}^{\infty} f(u)\mathcal{F}(g)(w) e^{-iwu}\,du,$$

which gives $\sqrt{2\pi}\mathcal{F}(f \star g) = 2\pi\mathcal{F}(f) \cdot \mathcal{F}(g)$. □

2.6.2. *From Parseval identity to Parseval formula (*TBA).*

2.6.3. *Fourier transform of distributions* $(*TBA)$.

3. Partial differential equations

3.1. **Functions of several variables.** In Laplace transform and Fourier transform, the functions e^{-st}, e^{-iwx} depend on two variables. We can look at e^{-st} as a map, say

$$f : (s,t) \mapsto f(s,t) := e^{-st},$$

from $\mathbb{R}^2$ to $\mathbb{R}$. We say that the map f defines a function on $\mathbb{R}^2$. It is clear that

$$f(x,y) = x^2 + y^2,$$

is a function on $\mathbb{R}^2$;

$$f(x,y,z) = x + y + z,$$

is a function on $\mathbb{R}^3$ and

$$f(t,x) = \frac{1}{\sqrt{2\pi t}}\, e^{-\frac{x^2}{2t}},$$

is a function on $(0,\infty) \times \mathbb{R}$.

3.1.1. *Graph.* If f is a function on $U \subset \mathbb{R}^n$ then we call the following set

$$G_{f,U} := \{(x, f(x)) \in \mathbb{R}^{n+1} : x \in U\},$$

in $\mathbb{R}^{n+1}$ the graph of f over U.

Exercise: Draw the graph of $f(x,y) = x^2 + y^2$ over the unit disc.

3.1.2. *Partial derivatives.* x-partial derivative of $f(x,y)$ means derivative of f with y fixed:

$$\frac{\partial f}{\partial x}(x,y) := \lim_{h\to 0}\frac{f(x+h,y)-f(x,y)}{h},$$

we write

$$f_x := \frac{\partial f}{\partial x}, \ \ f_{xy} := \frac{\partial f_x}{\partial y}.$$

Example: If $f(x,y)=x^2+y^2+xy$ then

$$f_x = 2x+y, \ \ f_y = 2y+x, \ f_{xy}=1=f_{yx}, \ f_{xx}=2, \ f_{yy}=2.$$

Example: If

$$f(t,x)=\frac{1}{\sqrt{2\pi t}}\, e^{-\frac{x^2}{2t}},$$

then

$$f_t = \frac{1}{\sqrt{2\pi}}\cdot\frac{-1}{2}\cdot t^{\frac{-3}{2}}\cdot e^{-\frac{x^2}{2t}} + \frac{1}{\sqrt{2\pi t}}\, e^{-\frac{x^2}{2t}}\cdot\frac{-x^2}{2}\cdot\frac{-1}{t^2} = \frac{x^2-t}{2t^2}f,$$

and

$$f_x = -\frac{x}{t}f, \ f_{xx} = -\frac{1}{t}f+\frac{x^2}{t^2}f = \frac{x^2-t}{t^2}f.$$

Thus we get that

$$f_t = \frac{1}{2}f_{xx}. \tag{11}$$

3.1.3. *Directional derivative and gradient.* Let $\mathbf{n}$ in the unit sphere be a given *direction*, then we call

$$f_{\mathbf{n}}(p) := \lim_{h\to 0}\frac{f(p+h\mathbf{n})-f(p)}{h},$$

the derivative of f along direction $\mathbf{n}$ at p. If we write

$$\mathbf{n}=(a,b,c), \ \ p=(x,y,z),$$

then

$$f_{\mathbf{n}}(p) = \lim_{h\to 0}\frac{f(x+ah,y+bh,z+ch)-f(x,y,z)}{h}.$$

Example: If

$$f(x,y,z)=c_0+c_1x+c_2y+c_3z,$$

then

$$f(h\mathbf{n}) = f(ah,bh,ch) = c_0+(c_1a+c_2b+c_3c)h,$$

which gives

$$f_{\mathbf{n}}(0) = c_1a+c_2b+c_3c = (f_x(0),f_y(0),f_z(0))\cdot\mathbf{n}. \tag{12}$$

Definition 3.1. *We call*

$$\nabla f(p) := (f_{x_1}(p),\cdots,f_{x_m}(p)),$$

the gradient of $f(x_1,\cdots,x_m)$ *at* p.

For a general smooth function, we have the following generalization of (12):

$$f_{\mathbf{n}}(p) = \nabla f(p) \cdot \mathbf{n}. \tag{13}$$

Since $|\mathbf{n}| = 1$, the above formula gives

$$f_{\mathbf{n}}(p) = \nabla f(p) \cos\theta,$$

where θ denotes the angle from $\nabla f(p)$ to $\mathbf{n}$.

3.1.4. *PDEs.* We shall study the following partial differential equations

1. *Wave equation*

$$u_{tt} = u_{xx}.$$

2. *Heat equation*

$$u_t = \frac{1}{2} u_{xx},$$

Homework: Find the background of the above two equations in [2] or wikipedia.

Example: By (11), we know that

$$f(t,x) := \frac{1}{\sqrt{2\pi t}}\, e^{-\frac{x^2}{2t}},$$

satisfies the above heat equation. We call $f(t,x)$ the *heat kernel*.

3.2. **Solve wave equation by Fourier series.** Let us solve the wave equation

$$u_{tt} = u_{xx},$$

with boundary conditions

$$u(t,0) = u(t,\pi) = 0, \ \ \forall\, t \geq 0;$$

and initial conditions

$$u(0,x) = f(x), \ u_t(0,x) = g(x), \ \ \forall\, 0 \leq x \leq \pi.$$

Step 1: Separating variables: Find solutions of the form

$$u(t,x) = G(t)F(x),$$

Since

$$u_{tt} = G''F, \ \ u_{xx} = GF'',$$

our equation becomes

$$G''F = GF'',$$

thus

$$\frac{G''}{G} = \frac{F''}{F} \equiv k,$$

where k is constant (notice that k does not depend on t and x).

Step 2: Fit boundary conditions: Notice that the boundary conditions

$$G(t)F(0) = G(t)F(\pi) = 0,$$

is equivalent to

$$F(0) = F(\pi) = 0.$$

In case $k = 0$, then $F'' \equiv 0$, i.e $F(x) = ax + b$. The boundary conditions give $F \equiv 0$.
In case $k = \mu^2 > 0$, then the general solution for

$$F'' = \mu^2 F,$$

is $F = Ae^{\mu x} + Be^{-\mu x}$, then the boundary conditions give

$$A + B = 0, \ \ Ae^{\mu\pi} + Be^{-\mu\pi} = 0,$$

thus $A = B = 0$.

Thus the only possible case is $k = -p^2 < 0$, then the general solution for

$$F'' = -p^2 F,$$

is $F = A\cos px + B\sin px$, $F(0) = 0$ gives $A = 0$. Thus $F = B\sin px$, $B \neq 0$, but $F(\pi) = 0$ gives $\sin p\pi = 0$, i.e.

$$p = n, \ \ n = 1, 2, \cdots,$$

(notice that $\sin -px = -\sin px$, thus up to a constant they give the same solution).

Summary: The boundary condition implies that $p = n^2$, $n = 1, 2, \cdots$, and

$$F = F_n(x) = \sin nx.$$

Now let us solve

$$G'' = -n^2 G,$$

the general solution is

$$G_n(t) = B_n \cos nt + C_n \sin nt.$$

Now we know that each

$$u_n(t, x) = G_n(t)F_n(t) = (B_n \cos nt + C_n \sin nt)\sin nx,$$

satisfies the wave equation and the boundary conditions, so does

$$u(t, x) = \sum_{n=1}^{\infty}(B_n \cos nt + C_n \sin nt)\sin nx.$$

Step 3: Fit the initial conditions: Choose B_n and C_n such that

$$u(0, x) = f(x), \ \ u_t(0, x) = g(x).$$

i.e.

$$\sum_{n=1}^{\infty} B_n \sin nx = f(x);$$

and

$$\sum_{n=1}^{\infty} nC_n \sin nx = g(x);$$

Consider odd extension f_o, g_o of f, g, by Theorem 2.6, if f_o and g_o is smooth enough, then it is enough to choose

$$B_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\, dx,$$

and

$$nC_n = \frac{2}{\pi}\int_0^{\pi} g(x)\sin nx\, dx.$$

Now we know that if f, g have smooth enough odd extension then

$$u(t, x) = \sum_{n=1}^{\infty}(B_n \cos nt + C_n \sin nt)\sin nx,$$

solves the wave equation (of course we have it check that it converges).

Example: When $g = 0$ we have

$$u(t,x) = \sum_{n=1}^{\infty} B_n \cos nt \sin nx.$$

Thus we can write

$$u(t,x) = \frac{1}{2}\sum_{n=1}^{\infty}(B_n \sin n(x-t) + B_n \sin n(x+t)) = \frac{1}{2}(f_o(x-t) + f_o(x+t)),$$

i.e. $u(t,x)$ is the superposition of two travelings of the initial wave.

Exercise: In case

$$f(x) = x,\ \ 0 \le x \le \frac{\pi}{2},\ \ f(x) = \pi - x,\ \ \frac{\pi}{2} < x \le \pi,$$

try to draw the graph of $u(t,x)$ for $t = 0, \frac{\pi}{8}, \frac{\pi}{4}, \frac{\pi}{2}, \pi$.

3.3. **Solve heat equation by Fourier series.** Consider the heat equation

$$u_t = \frac{1}{2}u_{xx},$$

with boundary conditions

$$u(t,0) = u(t,\pi) = 0,\ \ \forall\, t \ge 0;$$

and initial conditions

$$u(0,x) = f(x),\ \ \forall\, 0 \le x \le \pi.$$

Step 1: Separating variables: Find solutions of the form

$$u(t,x) = G(t)F(x),$$

Since

$$u_t = G'F,\ \ u_{xx} = GF'',$$

our equation becomes

$$G'F = \frac{1}{2}GF'',$$

thus

$$\frac{2G'}{G} = \frac{F''}{F} \equiv k,$$

Step 2: Fit the boundary conditions: Same as the wave equation, we have

$$k = -n^2,\ \ n = 1, 2 \cdots, n$$

and

$$F_n(x) = \sin nx.$$

Then

$$G' = -\frac{n^2}{2}G,$$

gives

$$G_n(t) = B_n e^{-\frac{n^2}{2}t},$$

Thus the general solution is

$$u(t,x) = \sum_{n=1}^{\infty} B_n e^{-\frac{n^2}{2}t} \sin nx.$$

Step 3: Fit the initial conditions: We hope

$$u(0,x)=\sum_{n=1}^{\infty} B_n \sin nx = f(x).$$

By Theorem 2.6, we get

$$B_n=\frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\, dx,$$

Example: $f(x)=2\sin x$ then

$$B_1=2,\ \ B_2=\cdots=B_n=0.$$

Thus

$$u(t,x)=2e^{-\frac{t^2}{2}}\sin x,$$

Notice that u goes to zero as t goes to infinity.

Exercise: Find u with

$$f(x)=x,\ \ 0\le x\le \frac{\pi}{2},\ \ f(x)=\pi-x,\ \ \frac{\pi}{2}\le x\le \pi.$$

3.4. **Solve heat equation by Fourier transform.** Still the heat equation

$$u_t=\frac{1}{2}u_{xx}.$$

But this time we consider the initial condition for all x in $\mathbb{R}$ (thus no boundary conditions):

$$u(0,x)=f(x),\ \ \forall -\infty<x<\infty.$$

Step 1: Reduce to ODE by Fourier transform: Consider Fourier transform of u_t with respect to the x variable

$$\mathcal{F}(u_t)=\widehat{u_t}(w)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} u_t(t,x)e^{-ixw}\, dx.$$

Then we have

$$\mathcal{F}(u_t)=\mathcal{F}(\frac{1}{2}u_{xx})=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{1}{2}u_{xx}(t,x)e^{-ixw}\, dx.$$

Recall that if u_x is smooth and rapidly decreasing then

$$\int_{-\infty}^{\infty} u_{xx}e^{-ixw}\, dx=\int_{-\infty}^{\infty} e^{-ixw}\, d(u_x)=-\int_{-\infty}^{\infty} u_x\, d(e^{-ixw})=iw\int_{-\infty}^{\infty} u_x e^{-ixw}\, dx,$$

the same computation for u gives

$$\int_{-\infty}^{\infty} u_x e^{-ixw}\, dx=iw\int_{-\infty}^{\infty} u e^{-ixw}\, dx,$$

thus we have

$$\mathcal{F}(u_t)=\frac{-w^2}{2}\mathcal{F}(u),\ \ \mathcal{F}(u):=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} u(t,x)e^{-ixw}\, dx$$

Notice that we also have

$$\mathcal{F}(u_t)=(\mathcal{F}(u))_t.$$

Thus $\mathcal{F}(u)$ satisfies the following ODE.

$$(\mathcal{F}(u))_t=\frac{-w^2}{2}\mathcal{F}(u).$$

Step 2: Solve the ODE and fit the initial condition: The general solution is

$$\mathcal{F}(u)(t,w) = c(w)e^{\frac{-w^2}{2}t}.$$

Notice that our initial condition implies

$$\mathcal{F}(u)(0,w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{-ixw}\,dx = \mathcal{F}(f).$$

Thus

$$c(w) = \mathcal{F}(f).$$

Now we have

$$\mathcal{F}(u)(t,w) = \mathcal{F}(f) \cdot e^{\frac{-w^2}{2}t}.$$

Step 3: use Fourier convolution formula: Recall that (see (6))

$$e^{\frac{-u^2}{2}} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{-y^2}{2}} e^{-iyu}\,dy.$$

Take

$$u = w\sqrt{t},\ y = \frac{x}{\sqrt{t}},$$

we get

$$e^{\frac{-w^2}{2}t} = \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} e^{\frac{-x^2}{2t}} e^{-ixw}\,dx = \frac{1}{\sqrt{t}}\mathcal{F}(e^{\frac{-x^2}{2t}}).$$

Thus

$$\mathcal{F}(u)(t,w) = \mathcal{F}(f) \cdot \left(\frac{1}{\sqrt{t}}\mathcal{F}(e^{\frac{-x^2}{2t}})\right)$$

Now the Fourier convolution formula gives

$$u(t,x) = \frac{1}{\sqrt{2\pi t}}(f \star e^{\frac{-x^2}{2t}}) = \int_{-\infty}^{\infty} f(p)\frac{1}{\sqrt{2\pi t}}e^{\frac{-(x-p)^2}{2t}}\,dp,$$

Summary: *the solution $u(t,x)$ is given by convolution of the initial temperature distribution with the heat kernel.*

4. Appendix: Definition of e, π and Euler's formula

4.1. **Where does e come from ?** Recall that: Let $A : \mathbb{C}^n \to \mathbb{C}^n$ be a linear map (here linear map means $A(au+bv) = aA(u)+bA(v)$ for all a, b in $\mathbb{C}$ and all u, v in $\mathbb{C}^n$). We call $u \neq 0$ in $\mathbb{C}^n$ an *eigenvector* of A if

$$Au = \lambda u, \tag{14}$$

where λ is a constant in $\mathbb{C}$.

What is an eigenvector of the derivative ?

By (14), we want to find function $u : \mathbb{R} \to \mathbb{C}$ such that

$$u' = \lambda u.$$

Power series method: Assume that

$$u(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots.$$

The following lemma gives:

$$u'(x) = a_1 + 2a_2x + \cdots + na_nx^{n-1} + (n+1)a_{n+1}x^n + \cdots.$$

Lemma 4.1. $(x^n)' = nx^{n-1},\ n = 1, 2, \cdots$.

Proof. If $n = 1$ then

$$x'(x) = \lim_{\triangle x \to 0} \frac{(x + \triangle x) - x}{\triangle x} = 1.$$

Assume the Lemma for $n = 1, \cdots, N - 1$. Then $(fg)' = f'g + fg'$ gives

$$(x^N)' = (x^{N-1})' \cdot x + x^{N-1} \cdot x' = (N-1)x^{N-2} \cdot x + x^{N-1} = Nx^{N-1}.$$

The proof is complete. □

Exercise: Why we have $(fg)' = f'g + fg'$?

Now

$$u' = \lambda u \Leftrightarrow \lambda a_n = (n+1)a_{n+1},\ n = 0, 1, \cdots .$$

Thus

$$a_{n+1} = \frac{\lambda a_n}{(n+1)} = \frac{\lambda^2 a_{n-1}}{(n+1)n} = \cdots = \frac{\lambda^{n+1} a_0}{(n+1)n \cdots 1} = \frac{\lambda^{n+1} a_0}{(n+1)!},$$

where we define

$$n! = 1 \cdot 2 \cdots n.$$

Then we have

$$u(x) = u_0 \cdot (1 + \lambda x + \cdots + \frac{(\lambda x)^n}{n!} + \cdots).$$

Put

$$E(x) := 1 + x + \cdots + \frac{x^n}{n!} + \cdots .$$

Since for every $C > 0$,

$$\lim_{n \to \infty} \frac{C^n}{n!} = 0,$$

we know that $E(x)$ converges for all x in $\mathbb{C}$.

Theorem 4.2. *$E(\lambda x)$ is a unique solution of the eigenvalue equation*

$$u' = \lambda u,$$

with initail condition $u(0) = 1$.

Definition 4.3. *We shall define*

$$e := E(1) = 1 + 1 + \frac{1}{2} + \cdots + \frac{1}{n!} + \cdots .$$

4.2. **Definition of the exponential function.** Let us write

$$e^2 = e \cdot e,\ e^3 = e^2 \cdot e,$$

and define e^m inductively by

$$e^{n+1} = e^n \cdot e.$$

Since e is positive, we can take the $q - th$ root of e^m, we write it as $e^{\frac{m}{q}}$. Thus for every $x \in \mathbb{Q}$, e^x is well defined. The following lemma tells us that $E(x)$ is an extension of e^x from $\mathbb{Q}$ to $\mathbb{C}$.

Lemma 4.4. *For every $x \in \mathbb{Q}$, we have $e^x = E(x)$.*

Proof. Since $E(1) = e$, it suffices to prove

$$E(\lambda_1)E(\lambda_2) = E(\lambda_1 + \lambda_2), \tag{15}$$

for every λ_1, λ_2 in $\mathbb{C}$. Notice that

$$(E(\lambda_1 x)E(\lambda_2 x))' = E(\lambda_1 x)'E(\lambda_2 x) + E(\lambda_2 x)'E(\lambda_1 x).$$

Put

$$G(x) = E(\lambda_1 x)E(\lambda_2 x).$$

Apply $E(\lambda x)' = \lambda E(\lambda x)$, we get

$$G' = (\lambda_1 + \lambda_2)G.$$

Notice that $G(0) = 1$. Thus Theorem 4.2 implies that

$$G(x) = E((\lambda_1 + \lambda_2)x).$$

Take $x = 1$, we get $E(\lambda_1)E(\lambda_2) = E(\lambda_1 + \lambda_2)$. □

Exercise: Find a direct proof of $E(\lambda_1)E(\lambda_2) = E(\lambda_1 + \lambda_2)$ without using Theorem 4.2.

Definition 4.5. *We shall use the same symbol e^x to denote $E(x)$ for all x in $\mathbb{C}$ and call e^x the **exponential function**. If $x > 0$ then we define $\ln x$ as the unique real solution of $e^{\ln x} = x$.*

By Theorem 4.2, we know that e^x is fully determined by

$$(e^x)' = e^x, \ e^0 = 1.$$

4.3. **Definition of π and trigonometric functions.** : Fix $P_0 = (1, 0)$ in the unit circle

$$S^1 := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}.$$

A counterclockwise rotation of P_0 gives a arc P_0P. The length, say $\theta(P)$, of the arc P_0P is a function of P. It is clear that the circumference diameter ratio is equal to $\theta(-1, 0)$.

Definition 4.6 (Definition of π)**.** *We shall write the circumference diameter ratio as π.*

Denote by

$$F : \theta(P) \mapsto P,$$

the inverse function of $0 \le \theta(P) \le 2\pi$.

Definition 4.7. *We shall write $F(\theta) = (\cos\theta, \sin\theta)$.*

Notice that

$$F(0) = (1, 0) = F(2\pi), \ F(\pi) = (-1, 0), \ |F(\theta)| \equiv 1.$$

In particular, it gives

$$\sin(0) = \sin(2\pi) = 0, \ \cos(0) = \cos(2\pi) = 1.$$

By definition of θ, we have

$$\int_0^{\hat\theta} |F'(\theta)| \, d\theta = \hat\theta, \ 0 \le \hat\theta \le 2\pi,$$

which gives

$$|F'(\theta)| \equiv 1.$$

Now $F(\theta) \cdot F(\theta) \equiv 1$ implies

$$F' \cdot F + F \cdot F' = 2F \cdot F' \equiv 0.$$

Hence $F' \perp F$, thus we know that

$$F'(\theta) = (-\sin\theta, \cos\theta), \text{ or } F'(\theta) = (\sin\theta, -\cos\theta).$$

But notice that $F'(0) = (0, 1)$, thus we must have

$$F'(\theta) = (-\sin\theta, \cos\theta),$$

which is equivalent to

$$(\cos\theta + i\sin\theta)' = i(\cos\theta + i\sin\theta).$$

Notice that $\cos 0 + i \sin 0 = 1$, thus Theorem 4.2 gives

Theorem 4.8 (Euler's formula). $e^{i\theta} = \cos\theta + i\sin\theta$.

Take $\theta = \pi$, we get the following Euler's identity

$$e^{i\pi} = -1.$$

Moreover, apply (15), we get

$$e^{i\theta_1} e^{i\theta_2} = e^{i(\theta_1+\theta_2)},$$

thus by Euler's formula, we have

$$(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) = \cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2),$$

i.e.

$$\cos(\theta_1+\theta_2) = \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2, \tag{16}$$

and

$$\sin(\theta_1+\theta_2) = \sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2. \tag{17}$$

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