These notes for TMA4135 (first seven weeks) are based on Erwin Kreyszig’s book [2], Dag Wessel-Berg’s video: http://video.adm.ntnu.no/serier/4fe2d4d3dbe03, and references [1, 3, 4] (among them [1] is very short and readable).

CONTENTS

1. Laplace transform 1
   1.1. Basic facts 1
   1.2. Laplace transform of derivatives, ODEs 2
   1.3. More Laplace transforms 3
2. Fourier analysis 9
   2.1. Complex and real Fourier series (Morten will probably teach this part) 9
   2.2. Fourier Sine and Cosine series 13
   2.3. Parseval’s identity 14
   2.4. Fourier transform 15
   2.5. Fourier inversion formula 16
   2.6. Fourier transform of derivative and convolution 18
3. Partial differential equations 19
   3.1. Functions of several variables 19
   3.2. Solve wave equation by Fourier series 21
   3.3. Solve heat equation by Fourier series 23
   3.4. Solve heat equation by Fourier transform 24
4. Appendix: Definition of $e$, $\pi$ and Euler’s formula 25
   4.1. Where does $e$ come from? 25
   4.2. Definition of the exponential function 26
   4.3. Definition of $\pi$ and trigonometric functions 27
References 28

I. LAPLACE TRANSFORM

1.1. Basic facts.

Definition 1.1. Let $f(t), t \geq 0$ be a given function. We call

$$F(s) := \int_{0}^{\infty} e^{-st} f(t) \, dt,$$

the Laplace transform of $f(t)$ and write

$$F = \mathcal{L}(f), \quad f = \mathcal{L}^{-1} F.$$

Date: August 16, 2018.
Remark: One can prove that the Laplace transform $\mathcal{L}$ is injective (see page 9 in [1]), that is the reason why $\mathcal{L}^{-1}$ is well defined (for a precise formula of $\mathcal{L}^{-1}$, see page 10 in [1]). To compute Laplace transforms, we need:

\[ d(fg) = f dg + g df, \int_a^b df = f(b) - f(a), \]

where $df := f'(t)dt$.

**Example 1:**

\[ \mathcal{L}(1) = \frac{1}{s}, \quad \mathcal{L}^{-1}\left(\frac{1}{s}\right) = 1, \quad s > 0. \]

**Example 2:**

\[ \mathcal{L}(e^{kt}) = \frac{1}{s - k}, \quad \mathcal{L}^{-1}\left(\frac{1}{s - k}\right) = e^{kt}, \quad s > k. \]

**Example 3:** $\mathcal{L}(e^{t^2})$ does not exist, 
\[
\int_0^\infty e^{-st}e^{t^2} \, dt = \infty,
\]
for all real number $s$.

**Remark:** Laplace transform is linear: By linearity, we mean for all real numbers $a, b$,

\[ \mathcal{L}(af + bg) = a\mathcal{L}(f) + b\mathcal{L}(g). \]

**Application 1:**

\[ \mathcal{L}(3 + 2e^{5t}) = 3\mathcal{L}(1) + 2\mathcal{L}(e^{5t}) = \frac{5(s - 3)}{s - 5}, \quad s > 5. \]

**Application 2:** Since 
\[
\frac{1}{s^2 - 3s + 2} = \frac{1}{(s - 1)(s - 2)} = \frac{1}{s - 2} - \frac{1}{s - 1},
\]
linearity gives 
\[
\mathcal{L}^{-1}\left(\frac{1}{s - 2}\right) = \mathcal{L}^{-1}\left(\frac{1}{s - 1}\right) = e^{2t} - e^t.
\]

**Proposition 1.2.** \(\mathcal{L}(t^n) = \frac{n!}{s^{n+1}}, \text{ for } n = 1, 2, \cdots, \text{ and } s > 0.\)

1.2. Laplace transform of derivatives, ODEs.

**Theorem 1.3** (Laplace transform of the derivative).
\[ \mathcal{L}(f') = s\mathcal{L}(f) - f(0). \]

**Example:** Solve: 
\[ y' = y, \quad y(0) = 1. \]

The answer is 
\[ y(t) = e^t. \]

**Remark:** Apply the theorem to $f'$, we get 
\[
\mathcal{L}(f'') = s\mathcal{L}(f') - f'(0) = s(s\mathcal{L}(f) - f(0)) - f'(0) = s^2\mathcal{L}(f) - sf(0) - f'(0).
\]
**Remark:** Apply the Laplace transform to a differential equation

\[ y'' + ay' + by = c(t), \quad a, b \in \mathbb{R}, \]

then we get

\[ s^2Y - sy(0) - y'(0) + a(sY - y(0)) + bY = C, \]

i.e

\[(s^2 + as + b)Y = (s + a)y(0) + y'(0) + C, \]

thus the inverse transform gives the solution

\[ y = \mathcal{L}^{-1}(Y) = \mathcal{L}^{-1}\left(\frac{(s + a)y(0) + y'(0) + C}{s^2 + as + b}\right). \]

**Example:** Consider

\[ y'' + 4y' + 4y = 0, \quad y(0) = 0, y'(0) = 1. \]

then the above formula gives

\[ y = \mathcal{L}^{-1}\left(\frac{1}{s^2 + 4s + 4}\right) = \mathcal{L}^{-1}\left(\frac{1}{(s + 2)^2}\right). \]

How to compute the inverse Laplace transform of \( \frac{1}{(s+2)^2} \)? Is it related to \( \mathcal{L}^{-1}\left(\frac{1}{s^2}\right) = t \)? We will answer them in the next section.

### 1.3. More Laplace transforms.

#### 1.3.1. \( s \)-Shifting.

The following formula

\[ \int_0^\infty e^{-st}e^{at}f(t)\,dt = F(s-a), \quad F(s), \]

gives

**Theorem 1.4** (\( s \)-Shifting theorem).

\[ \mathcal{L}(e^{at}f(t)) = F(s-a). \]

**Example:**

\[ \mathcal{L}^{-1}\left(\frac{1}{(s + 2)^2}\right) = e^{-2t}. \]

**Example:**

\[ \mathcal{L}(e^{kt}) = \frac{1}{s-k}. \]

Take \( k = iw \),

\[ \mathcal{L}(e^{iwt}) = \frac{1}{s-\text{i}w} = \frac{s + \text{i}w}{s^2 + w^2}. \]

Euler’s formula (see the appendix) gives

(5) \[ \mathcal{L}(\cos wt) = \frac{s}{s^2 + w^2}, \quad \mathcal{L}(\sin wt) = \frac{w}{s^2 + w^2}. \]

**Example:**

\[ \mathcal{L}^{-1}\left(\frac{s + 2}{s^2 + 4}\right) = \mathcal{L}^{-1}\left(\frac{s}{s^2 + 4}\right) + \mathcal{L}^{-1}\left(\frac{2}{s^2 + 4}\right) = \cos 2t + \sin 2t. \]

**Exercises:**
1. Find the inverse Laplace transform of \( \frac{s}{s^2 + 2s + 2} \). The answer is
\[
\mathcal{L}^{-1}\left(\frac{s}{s^2 + 2s + 2}\right) = e^{-t}(\cos t - \sin t).
\]

2. Solve \( y'' - y = t, \ y(0) = y'(0) = 1 \). The answer is
\[
y = e^t + \frac{1}{2}(e^t - e^{-t}) - t.
\]

**Definition 1.5** (\( \sinh t \) and \( \cosh t \)).

\[
\sinh t := \frac{e^t - e^{-t}}{2}, \quad \cosh t := \frac{e^t + e^{-t}}{2}.
\]

**Exercises:**

3. Compute Laplace transform of \( \sinh t \) and \( \cosh t \). The answer is
\[
\mathcal{L}(\sinh t) = \frac{1}{s^2 - 1}, \quad \mathcal{L}(\cosh t) = \frac{s}{s^2 - 1}.
\]

4. Compute Laplace transform of
\[
f(t) = 1 \text{ if } 3 < t < 4; \quad f(t) = 0 \text{ otherwise}.
\]
The answer is
\[
\mathcal{L}(f) = \frac{e^{-3s} - e^{-4s}}{s}.
\]

1.3.2. **Laplace transform of integrals.** Put
\[
g(t) = \int_0^t f(\tau) \, d\tau,
\]
then
\[
g'(t) = f(t), \quad g(0) = 0.
\]
Thus
\[
F = \mathcal{L}(f) = \mathcal{L}(g') = sG - g(0) = sG
\]
gives \( G = \frac{F}{s} \), i.e.
\[
\mathcal{L}\left(\int_0^t f(\tau) \, d\tau\right) = \frac{\mathcal{L}(f)}{s}.
\]

**Exercises:**

1. Show that
\[
\mathcal{L}^{-1}\left(\frac{1}{s(s^2 + 1)}\right) = 1 - \cos t.
\]

2. Show that
\[
\mathcal{L}^{-1}\left(\frac{1}{s} \cdot \frac{1}{s(s - 1)}\right) = e^t - 1 - t.
\]
1.3.3. Using step functions.

**Definition 1.6** (Step function). Let \( a \geq 0 \), the step function \( u(t - a) \) is defined as follows

\[
\begin{align*}
    u(t - a) &= 0, & \text{for } 0 \leq t < a; \\
    u(t - a) &= 1, & \text{for } t \geq a.
\end{align*}
\]

In case \( a = 0 \) we call \( u(t) \) the Heaviside function.

**Exercise**:

1. Draw the graphs of
   \[
   f(t) = u(t - 1) - u(t - 3),
   \]
   \[
   f(t) = (u(t) - u(t - \pi)) \sin t,
   \]
   \[
   f(t) = u(t) + u(t - 1) + \cdots + u(t - n) + \cdots.
   \]

2. Show that
   \[
   \mathcal{L}(u(t - a)) = \frac{e^{-as}}{s}.
   \]

3. Compare the graphs of \( u(t - a) f(t - a) \) with that of \( f(t) \).

**Theorem 1.7** \((t\text{-Shifting theorem})\).

\[
\mathcal{L}(u(t - a)f(t - a)) = e^{-as}\mathcal{L}(f).
\]

**Example**: Since

\[
\frac{1}{s - 2} = \mathcal{L}(e^{2t}),
\]

we have

\[
\mathcal{L}^{-1}(e^{-s}s^{-1}) = u(t - 1)e^{2(t-1)}.
\]

**Exercise**:

1. Compute Laplace transform of
   \[
   f(t) = \sum_{n=0}^{\infty} u(t - n).
   \]
   (Hint: \( \sum_{n=0}^{\infty} r^n = \frac{1}{1-r} \) for \( |r| < 1 \)) Answer
   \[
   \mathcal{L}\left(\sum_{n=0}^{\infty} u(t - n)\right) = \frac{1}{s(1 - e^{-s})}.
   \]

**RC-Circuit equation** (see page 29 section 1.5 and page 93 section 2.9 of Kreyszig’s book): \( R, C \) positive constants, \( i(t), e(t) \) functions:

\[
Ri(t) + \frac{1}{C} \int_{0}^{t} i(\tau) d\tau = e(t).
\]

Apply the Laplace transform, we get

\[
RI(s) + \frac{1}{C} \cdot \frac{I(s)}{s} = E(s),
\]
i.e.

\[ I(s) = \frac{E(s)}{R + \frac{1}{vs}} = \frac{s}{s + \frac{1}{\pi c}} \frac{E(s)}{R}. \]

**Exercise:**

2. Compute \( i(t) \) when

\[ e(t) = u(t - 1) - u(t - 2), \quad R = C = 1. \]

Answer

\[ i(s) = u(t - 1)e^{-(t-1)} - u(t - 2)e^{-(t-2)}. \]

1.3.4. Dirac delta function. The delta function \( \delta (t - a) \) is defined by

\[ \int_0^\infty f(t)\delta(t - a) \, dt = f(a). \]

In case \( f(t) = e^{-st} \), we have

\[ \int_0^\infty e^{-st}\delta(t - a) \, dt = e^{-as}. \]

Thus

\[ \mathcal{L}(\delta(t - a)) = e^{-as}. \]

**Exercise:** solve

\[ y'' + y = \delta(t - 1), \quad y(0) = y'(0) = 0. \]

Answer

\[ y = \mathcal{L}^{-1}(e^{-s}\mathcal{L}(\sin t)) = u(t - 1) \sin(t - 1). \]

1.3.5. Convolution. Let \( f(t), g(t) \) be two functions for \( t \geq 0 \).

**Definition 1.8** (Convolution of \( f \) and \( g \)).

\[ (f \ast g)(t) := \int_0^t f(\tau)g(t - \tau) \, d\tau, \quad t \geq 0. \]

**Examples:**

\[ 1 \ast t = \frac{t^2}{2}, \]

\[ e^t \ast e^t = te^t, \]

\[ f(t) \ast 1 = \int_0^t f(\tau) \, d\tau. \]

**Theorem 1.9** (Laplace transform of convolution).

\[ \mathcal{L}(f \ast g) = \mathcal{L}(f) \cdot \mathcal{L}(g). \]
Exercises:

1. Compute $t^m \ast t^n$. Hint: use $t^m \ast t^n = \mathcal{L}^{-1}(t^m \ast t^n)$, answer
   
   $$
   t^m \ast t^n = \frac{m!n!}{(m+n+1)!} t^{m+n+1}, \quad m, n = 0, 1, \ldots
   $$

2. Compute $\mathcal{L}^{-1}\left(\frac{1}{(s^2+1)^2}\right)$. Hint: use $\mathcal{L}^{-1}\left(\frac{1}{(s^2+1)^2}\right) = \mathcal{L}^{-1}(\mathcal{L}(\sin t) \cdot \mathcal{L}(\sin t))$. Answer
   
   $$
   \mathcal{L}^{-1}\left(\frac{1}{(s^2+1)^2}\right) = \frac{\sin t - t \cos t}{2}.
   $$

3. Solve $y'' + y = \sin t$, $y(0) = 0$, $y'(0) = 1$. Hint: use Exercise 2. Answer
   
   $$
   y = \sin t + \frac{\sin t - t \cos t}{2} = \frac{3\sin t - t \cos t}{2}.
   $$

4. Solve: $y - \int_0^t (t - \tau) y(\tau) \, d\tau = 1$. Hint: use $\int_0^t (t - \tau) y(\tau) \, d\tau = y \ast t$. Answer
   
   $$
   y = \frac{e^t + e^{-t}}{2} = \cosh t.
   $$

1.3.6. Non-homogeneous linear ODEs. Consider

   $$
   y'' + by' + cy = r(t),
   $$

given $y(0)$ and $y'(0)$, we have

   $$
   s^2 Y - sy(0) - y'(0) + b(sY - y(0)) + cY = R(s).
   $$

Thus

   $$
   Y = \frac{1}{s^2 + bs + c} \cdot R(s) + \frac{sy(0) + y'(0) + by(0)}{s^2 + bs + c} := K(s) \cdot R(s) + G(s),
   $$

we get

   $$
   y = k \ast r + g.
   $$

Example: Consider

   $$
   y'' + y = r(t), \quad y(0) = y'(0) = 0.
   $$

Apply the Laplace transform, we have

   $$
   s^2 Y + Y = \mathcal{L}(r).
   $$

Thus

   $$
   Y = \frac{1}{s^2 + 1} \cdot \mathcal{L}(r),
   $$

which gives

   $$
   y(t) = \sin t \ast r.
   $$
1.3.7. Derivative of the Laplace transform. Apply differential to
\[ F(s) = \int_0^\infty e^{-st} f(t) \, dt, \]
we get
\[ F'(s) = \int_0^\infty \frac{d(e^{-st})}{ds} f(t) \, dt = \int_0^\infty e^{-st} \cdot (-tf(t)) \, dt = \mathcal{L}(-tf(t)). \]

**Example 1:**
\[ \mathcal{L}(t \sin t) = -\left( \frac{1}{s^2 + 1} \right)' = \frac{2s}{(s^2 + 1)^2}. \]

**Example 2:** Let \( F(s) = \ln(1 + s^{-2}) \). Then
\[ F' = (\ln(1 + s^2) - \ln(s^2))' = \frac{2s}{1 + s^2} - \frac{2}{s}. \]
Thus
\[ \mathcal{L}^{-1}(F') = 2 \cos t - 2. \]

By the above theorem, we have
\[ \mathcal{L}^{-1}(F') = -tf(t). \]

Thus
\[ f(t) = \mathcal{L}^{-1}(\ln(1 + s^{-2})) = \frac{2 - 2 \cos t}{t}. \]

1.3.8. System of differential equations. Look at this example:
\[ y_1' = -y_1 + y_2; \quad y_2' = -y_1 - y_2 + f(t), \quad y_1(0) = y_2(0) = 0. \]

Apply the Laplace transform, we get
\[ sY_1 = -Y_1 + Y_2; \quad sY_2 = -Y_1 - Y_2 + F(s). \]
Thus
\[ (s + 1)Y_1 - Y_2 = 0; \]
\[ Y_1 + (s + 1)Y_2 = F(s). \]
The first equation gives \( Y_2 = (s + 1)Y_1 \), together with the second, we have
\[ Y_1 = F(s)(1 + (s + 1)^2)^{-1}. \]

Use the first equation again,
\[ Y_2 = F(s)(s + 1)(1 + (s + 1)^2)^{-1}. \]
Thus
\[ y_1 = f(t) * (e^{-t} \sin t), \quad y_2 = f(t) * (e^{-t} \cos t). \]
when \( f(t) = e^{-t} \), we get
\[ y_1 = \int_0^t e^{-(t-\tau)} e^{-\tau} \sin \tau \, d\tau = e^{-t}(1 - \cos t), \quad y_2 = e^{-t} \sin t. \]
1.3.9. **Homework for Laplace transform.** Please compute Laplace transform of

1. \( f(t) = t, \) if \( 0 \leq t \leq a, \) \( f(t) = 0, \) if \( t > a. \) Answer

\[
F(s) = \frac{1}{s^2} - \frac{e^{-as}}{s^2} - a \frac{e^{-as}}{s}.
\]

2. \( f(t) = u(t - \pi) \sin t. \) Answer

\[
F(s) = -\frac{e^{-\pi s}}{s^2 + 1}.
\]

3. Solve the following equation

\[
i'(t) + 2i(t) + \int_0^t i(\tau) d\tau = \delta(t - 1), \quad i(0) = 0.
\]

Answer

\[
i(t) = u(t - 1)(e^{-(t-1)} - e^{-(t-1)}(t - 1)).
\]

### 2. Fourier Analysis

2.1. **Complex and real Fourier series (Morten will probably teach this part).**

2.1.1. **Complex Fourier series.** Fix \( p > 0, \) if \( f(x + p) = f(x), \) \( \forall x \in \mathbb{R}, \)

then call \( f \) a **periodic function with period** \( p. \)

**Example: periodic function:**

1. A polynomial is periodic if and only if it is a constant;
2. \( e^{\lambda x} \) has period \( 2\pi \) if and only if \( \lambda = in, \) \( n \in \mathbb{Z}. \)

The main theorem in Fourier analysis is the following:

**Theorem 2.1** (Fourier 1807). *If \( f \) has period \( 2\pi \) and is smooth enough then we have*

\[
f(x) = \sum_{n \in \mathbb{Z}} c_n e^{inx}, \quad \forall x \in \mathbb{R}.
\]

*—The proof (see Page 63 in [3]) is not assumed in this course.*

**What does "smooth enough" mean?** It means that \( f \) is piecewise smooth and

\[
f(x_0) = \frac{f(x_0^+) + f(x_0^-)}{2},
\]

if \( f \) is not smooth at \( x_0. \)

**How to compute \( c_n \)?** We shall prove that

\[
c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx.
\]

In fact, by the above theorem, we have

\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx = \sum_{m \in \mathbb{Z}} \frac{c_m}{2\pi} \int_{-\pi}^{\pi} e^{inx} e^{-inx} dx.
\]

If \( m = n \) then

\[
\int_{-\pi}^{\pi} e^{inx} e^{-inx} dx = \int_{-\pi}^{\pi} 1 dx = 2\pi.
\]
If $m \neq n$ then
\[ \int_{-\pi}^{\pi} e^{i(m-n)x} \, dx = \int_{-\pi}^{\pi} e^{i(m-n)x} i(m-n) \, dx = 0. \]
Thus
\[ \sum_{m \in \mathbb{Z}} \frac{c_m}{2\pi} \int_{-\pi}^{\pi} e^{imx} e^{-inx} \, dx = c_n. \]

**Definition 2.2.** We call
\[ f(x) = \sum_{n \in \mathbb{Z}} c_n e^{inx} \]
the **complex Fourier series** of $f$ and
\[ c_n := \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)e^{-inx} \, dx, \quad n \in \mathbb{Z}, \]
the **complex Fourier coefficients** of $f$.

**Example:** Consider
\[ f(x) = 1, \quad 0 < x < \pi; \quad f(x) = -1, \quad -\pi < x < 0, \]
and
\[ f(0) = f(\pi) = f(-\pi) = 0. \]
Then we know $f$ is smooth enough and
\[ 2\pi c_n = \int_{-\pi}^{\pi} f(x)e^{-inx} \, dx = \int_{0}^{\pi} e^{-inx} \, dx - \int_{-\pi}^{0} e^{-inx} \, dx. \]
Since
\[ \int_{0}^{\pi} e^{-inx} \, dx = \int_{0}^{\pi} \frac{e^{-inx}}{-in} \, dx = \frac{(-1)^n - 1}{-in}, \]
and
\[ \int_{-\pi}^{0} e^{-inx} \, dx = \int_{-\pi}^{0} \frac{e^{-inx}}{-in} \, dx = \frac{1 - (-1)^n}{-in}, \]
we have
\[ 2\pi c_n = \frac{2(1 - (-1)^n)}{in}, \]
i.e.
\[ c_n = \frac{2}{in\pi}, \quad n \text{ odd}; \quad c_n = 0, \quad n \text{ even}. \]
Thus the complex Fourier series of $f$ is
\[ f(x) = \sum_{m \in \mathbb{Z}} \frac{2}{i(2m+1)\pi} e^{i(2m+1)x}. \]
2.1.2. **(Real) Fourier series.** In the previous example, we have
\[ f(x) = \frac{2}{i\pi} \left( e^{ix} + e^{3ix} \frac{1}{3} + \cdots \right) + \frac{2}{i\pi} \left( \frac{e^{-ix}}{-1} + e^{-3ix} \frac{1}{3} + \cdots \right), \]

thus
\[ f(x) = \frac{2}{i\pi} \left( e^{ix} - e^{-ix} + \frac{3ix}{3} - e^{-3ix} + \cdots \right). \]

Euler’s formula gives
\[ e^{inx} - e^{-inx} = 2i \sin nx, \]

thus
\[ f(x) = \frac{4}{\pi} \left( \sin x + \frac{3x}{3} + \cdots \right), \]

In particular, it gives
\[ 1 = f\left(\frac{\pi}{2}\right) = \frac{4}{\pi} \left( 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots \right). \]

Thus
\[ 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}, \]

which is a famous formula obtained by Leibniz in 1673 from geometric considerations.

For a general function \( f \), by the Euler formula, we have
\[ f(x) = \sum c_n e^{inx} = \sum c_n (\cos nx + i \sin nx), \]

which gives
\[ f(x) = c_0 + \sum_{n=1}^{\infty} c_n (\cos nx + i \sin nx) + \sum_{n=1}^{\infty} c_{-n} (\cos nx - i \sin nx). \]

Thus we have
\[ f(x) = c_0 + \sum_{n=1}^{\infty} ((c_n + c_{-n}) \cos nx + i(c_n - c_{-n}) \sin nx), \]

Recall that
\[ (c_n + c_{-n}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)(e^{-inx} + e^{inx}) \, dx = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx, \]

and
\[ i(c_n - c_{-n}) = \frac{i}{2\pi} \int_{-\pi}^{\pi} f(x)(e^{-inx} - e^{inx}) \, dx = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx, \]

thus we get

**Theorem 2.3.** If \( f \) has period \( 2\pi \) and is smooth enough then it has the following **Fourier series** expansion
\[ f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx), \]

where \( a_0, a_n, b_n \) are the **Fourier coefficients** of \( f \) such that
\[ a_0 = c_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \, dx, \]
and for \( n = 1, 2 \cdots \), we have
\[
a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx,
\]
and
\[
b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx.
\]

**Example:** Consider
\[
f(x) = 0, \quad -\pi < x < 0; \quad f(x) = x, \quad 0 \leq x < \pi.
\]
Then
\[
2\pi a_0 = \int_{-\pi}^{\pi} f(x) \, dx = \int_{0}^{\pi} x \, dx = \frac{\pi^2}{2},
\]
and
\[
\pi a_n = \int_{-\pi}^{\pi} f(x) \cos nx \, dx = \int_{0}^{\pi} x \cos nx \, dx = \int_{0}^{\pi} x \, d\left(\frac{\sin nx}{n}\right) = -\int_{0}^{\pi} \frac{\sin nx}{n} \, dx.
\]
Since
\[
\int_{0}^{\pi} \frac{\sin nx}{n} \, dx = \int_{0}^{\pi} d\left(\frac{\cos nx}{n^2}\right) = \frac{(-1)^n - 1}{n^2},
\]
we get
\[
a_0 = \frac{\pi}{4}, \quad a_{2m} = 0, \quad a_{2m-1} = \frac{-2}{(2m-1)2\pi}, \quad m = 1, 2 \cdots .
\]
Moreover, we have
\[
\pi b_n = \int_{-\pi}^{\pi} f(x) \sin nx \, dx = \int_{0}^{\pi} x \sin nx \, dx = \int_{0}^{\pi} x \, d\left(\frac{-\cos nx}{n}\right) = \frac{\pi(-1)^{n+1}}{n} + \int_{0}^{\pi} \frac{\cos nx}{n} \, dx,
\]
Notice that
\[
\int_{0}^{\pi} \frac{\cos nx}{n} \, dx = \int_{0}^{\pi} d\left(\frac{\sin nx}{n^2}\right) = 0,
\]
thus
\[
b_n = \frac{(-1)^{n+1}}{n}.
\]
Thus
\[
f(x) = \frac{\pi}{4} - \frac{2}{\pi} \left(\cos x + \frac{\cos 3x}{3^2} + \cdots\right) + \left(\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \cdots\right).
\]
Take \( x = 0 \) then we get
\[
0 = \frac{\pi}{4} - \frac{2}{\pi} \left(1 + \frac{1}{3^2} + \cdots\right),
\]
i.e.
\[
1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}.
\]

**Exercise:** Use \( 1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8} \) to prove that
\[
1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}.
\]
Proposition 2.4. Put $\delta_{mn} = 1$ if $m = n$ and $\delta_{mn} = 0$ if $m \neq n$ then
\[
\int_{-\pi}^{\pi} \cos nx \cos mx \, dx = \int_{-\pi}^{\pi} \sin nx \sin mx \, dx = \pi \delta_{mn}, \quad m, n = 1, 2, \ldots,
\]
and
\[
\int_{-\pi}^{\pi} \cos nx \, dx = 2\pi \delta_{n0}, \quad \int_{-\pi}^{\pi} \cos nx \sin mx = 0, \quad m, n = 0, 1, 2, \ldots.
\]

Proof. Follows from the Euler formula
\[
\cos nx = \frac{e^{inx} + e^{-inx}}{2}, \quad \sin nx = \frac{e^{inx} - e^{-inx}}{2i},
\]
and
\[
\int_{-\pi}^{\pi} e^{inx}e^{-inx} \, dx = 2\pi \delta_{mn}.
\]
Try to give the details yourself. \hfill \square

Exercise: Try to use the above proposition to prove the formulas for $a_n, b_n$ in Theorem 2.3.

2.2. Fourier Sine and Cosine series.

Definition 2.5. We say that $f$ is odd if $f(-x) = -f(x)$; $f$ is even if $f(-x) = f(x)$.

Example: For every positive integer $n$, we know that $\cos nx$ is even and $\sin nx$ is odd.

Application: If $f$ is even then
\[
\int_{-\pi}^{\pi} f(x) \, dx = 2 \int_{0}^{\pi} f(x) \, dx.
\]
If $f$ is odd then
\[
\int_{-\pi}^{\pi} f(x) \, dx = 0.
\]
In particular, if $f$ is odd then all
\[
a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx = 0;
\]
if $f$ is even then all
\[
b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx = 0.
\]
Thus we get:

Theorem 2.6. Assume that $f$ has period $2\pi$ and is smooth enough. If $f$ is odd then it can be written as a Fourier sine series
\[
f(x) = \sum_{n=1}^{\infty} b_n \sin nx, \quad b_n = \frac{2}{\pi} \int_{0}^{\pi} f(x) \sin nx \, dx.
\]
If $f$ is even then it can be written as a Fourier cosine series
\[
f(x) = \frac{1}{\pi} \int_{0}^{\pi} f(x) \, dx + \sum_{n=1}^{\infty} a_n \cos nx, \quad a_n = \frac{2}{\pi} \int_{0}^{\pi} f(x) \cos nx \, dx.
\]
Odd or Even extension: Let \( f \) be a function in \((0, \pi)\). Then we can extend \( f \) to an odd function, say \( f_o \) such that 
\[
f_o(-x) = -f(x), \quad x \in (0, \pi);
\]
we can also extend \( f \) to an even function, say \( f_e \) such that 
\[
f_e(-x) = f(x), \quad x \in (0, \pi).
\]

Exercise:
1. Draw the graph of the odd extension \( f_o \) and the even extension \( f_e \) of the following function
\[
f(x) = x, \quad 0 < x < \frac{\pi}{2}; \quad f(x) = \frac{\pi}{2}, \quad \frac{\pi}{2} < x < \pi.
\]
2. Find the Fourier cosine series of \( f_e \). Answer
\[
f_e(x) = \frac{3\pi}{8} + \frac{2}{\pi} \left( -\cos x - \frac{2\cos 2x}{2^2} - \frac{\cos 3x}{3^2} - \frac{\cos 5x}{5^2} - \cdots \right).
\]
3. Find the Fourier sine series of \( f_o \). Answer
\[
f_o(x) = \left( \frac{2}{\pi} + 1 \right) \sin x + \left( 0 - \frac{1}{2} \right) \sin 2x + \left( \frac{-2}{3^2\pi} + \frac{1}{3} \right) \sin 3x
\]
\[
+ \left( 0 + \frac{1}{4} \right) \sin 4x + \left( \frac{2}{5^2\pi} + \frac{1}{5} \right) \sin 5x + \cdots.
\]

2.3. Parseval’s identity. Let \( f \) be a smooth enough function with period \( 2\pi \). Consider the complex Fourier series expansion of \( f \)
\[
f = \sum_{n \in \mathbb{Z}} c_n e^{inx}.
\]

We have
\[
\int_{-\pi}^{\pi} |f(x)|^2 \, dx = \int_{-\pi}^{\pi} \sum_{n,m \in \mathbb{Z}} c_n \overline{c_m} e^{inx-imx} \, dx.
\]

Now use that
\[
\int_{-\pi}^{\pi} e^{inx-imx} \, dx = 0,
\]
if \( m \neq n \) and
\[
\int_{-\pi}^{\pi} e^{inx-imx} \, dx = 2\pi,
\]
if \( m = n \). We get the following Parseval identity
\[
\int_{-\pi}^{\pi} |f(x)|^2 \, dx = 2\pi \sum_{n \in \mathbb{Z}} |c_n|^2.
\]

Example: Consider the last example in section 2.1.1:
\[
f(x) = 1, \quad 0 < x < \pi; \quad f(x) = -1, \quad -\pi < x < 0,
\]
and
\[
f(0) = f(\pi) = f(-\pi) = 0.
\]
We know that \( f \) has the following complex Fourier series expansion:
\[
f(x) = \sum_{m \in \mathbb{Z}} \frac{2}{i(2m+1)\pi} e^{i(2m+1)x}.
\]
Thus the Parseval identity gives
\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 \, dx = 1 = \sum_{m \in \mathbb{Z}} \frac{4}{(2m + 1)^2 \pi^2} = \frac{8}{\pi^2} \left( 1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots \right),
\]
which gives another proof of
\[
1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}.
\]

2.4. Fourier transform.

Definition 2.7. We call
\[
\hat{f}(w) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-iwx} \, dx,
\]
the Fourier transform of \(f\) and write \(\hat{f} = \mathcal{F}(f)\).

Example: F1: Fourier transform of \(f(x) = 1\) if \(|x| < 1\) and \(f(x) = 0\) otherwise:
\[
\hat{f}(w) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-iwx} \, dx = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} e^{-iwx} \, dx
\]
if \(w \neq 0\) then
\[
\int_{-1}^{1} e^{-iwx} \, dx = \int_{-1}^{1} d\left( \frac{e^{-iwx}}{-iw} \right) = \frac{e^{-iw} - e^{iw}}{-iw} = \frac{2\sin w}{w}.
\]
Notice that
\[
\lim_{w \to 0} \frac{2\sin w}{w} = 2 = \int_{-1}^{1} dx = \hat{f}(0).
\]
Thus we can write
\[
\hat{f}(w) = \frac{1}{\sqrt{2\pi}} \left( \frac{2\sin w}{w} \right) = \sqrt{\frac{2}{\pi}} \frac{\sin w}{w}.
\]

Example: F2: Fourier transform of \(f(x) = e^{-x}\) if \(x > 0\) and \(f(x) = 0\) otherwise:
\[
\hat{f}(w) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-iwx} \, dx = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-x} e^{-iwx} \, dx = \frac{1}{\sqrt{2\pi}} \mathcal{L}(e^{-iwt})(1).
\]
Recall that
\[
\mathcal{L}(e^{-iwt})(s) = \frac{1}{s + iw},
\]
thus
\[
\hat{f}(w) = \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{1 + iw}.
\]
2.4.1. From Complex Fourier series to inverse Fourier transform. Assume that $f$ is smooth enough in $-N < x < N$ and $f = 0$ when $|x| > N$. For each $L > N$, let us define a periodic function $f_L$ such that

$$f_L(x) = f(x), \; |x| < L; \; f_L(x + 2L) = f_L(x).$$

Then we know that $g_L(x) = f_L\left(\frac{\pi x}{L}\right)$, has period $p$ and is smooth enough. Thus

$$g_L(x) = \sum c_n e^{inx}, \; c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} g_L(x) e^{-inx} dx.$$ 

Thus

$$f_L(x) = g_L\left(\frac{\pi x}{L}\right) = \sum c_n e^{in\frac{\pi x}{L}}.$$ 

Consider $v = \frac{Lx}{\pi}$, we can write

$$c_n = \frac{1}{2\pi} \int_{-L}^{L} f_L(v) e^{-in\frac{\pi v}{L}} d\left(\frac{\pi v}{L}\right) = \frac{1}{2L} \int_{-\infty}^{\infty} f(v) e^{-in\frac{\pi v}{L}} dv = \frac{\sqrt{2\pi}}{2L} \hat{f}\left(n\frac{\pi L}{2}\right).$$ 

which gives

$$f(x) = \sqrt{\frac{\pi}{2}} \sum_{n \in \mathbb{Z}} \hat{f}\left(n\frac{\pi L}{2}\right) e^{in\frac{\pi x}{L}}.$$ 

Put $\Delta w = \frac{\pi}{L}$, then we have

$$f(x) = \frac{1}{\sqrt{2\pi}} \sum_{n \in \mathbb{Z}} \hat{f}(n\Delta w) e^{inx} \Delta w.$$ 

Assume that $\hat{f}(w) e^{ixw}$ is integrable in $-\infty < x < \infty$. Let $L$ goes to infty, the above formula gives the following Fourier inversion formula:

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(w) e^{ixw} dw.$$ 

We say that $f(x)$ is the inverse Fourier transform of $\hat{f}(w)$ and write $f = \mathcal{F}^{-1}(\hat{f})$.

2.5. Fourier inversion formula. When do we have the Fourier inversion formula? It is known that (see Page 141 Theorem 1.9 in [3]) the Fourier inversion formula is true if $f$ is smooth and rapidly decreasing, in the sense that

$$\sup_{x \in \mathbb{R}} |x|^k |f^{(l)}(x)| < \infty, \; \text{for every } k, l \geq 0,$$

where $f^{(l)}$ denotes the $l$-th derivative of $f$.

**Example:** Let $f(x) = e^{-\frac{x^2}{2}}$. We shall use Fourier inversion formula to prove

$$\hat{f}(w) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} e^{-iwx} dx = e^{-\frac{w^2}{2}} = f(w).$$

**Step 1:** Look at the derivative of $\hat{f}(w)$:

$$(\hat{f})'(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} (-ix) e^{-iwx} dx = \mathcal{F}(-ixf(x)).$$
Notice that \((e^{-\frac{x^2}{2}})' = e^{-\frac{x^2}{2}}(-x)\), thus

\[
\hat{f}'(w) = \frac{i}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ixw} d(e^{-\frac{x^2}{2}}) = \frac{-i}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} d(e^{-iwx}) = -w \hat{f}(w).
\]

Now we have

\[
\left( \hat{f}(w)e^{\frac{w^2}{2}} \right)' = (-w + w) \left( \hat{f}(w)e^{\frac{w^2}{2}} \right) = 0.
\]

Thus \(\hat{f}(w)e^{\frac{w^2}{2}}\) is a constant, i.e.

\[
\hat{f}(w)e^{\frac{w^2}{2}} \equiv \hat{f}(0)e^0 = \hat{f}(0).
\]

Now we have

\[
\hat{f}(w) = \hat{f}(0)e^{\frac{w^2}{2}} = \hat{f}(0)f(w)
\]

Step 2: The Fourier inversion formula implies (notice that \(f\) is smooth and rapidly decreasing)

\[
f(x) = \mathcal{F}^{-1}(\hat{f}) = \hat{f}(0)\mathcal{F}^{-1}(f) = \hat{f}(0)\hat{f}(-x) = (\hat{f}(0))^2f(x).
\]

Thus \(\hat{f}(0) = \pm 1\). Since

\[
\hat{f}(0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} dx > 0,
\]

we get

\[
\hat{f}(0) = 1, \quad \hat{f} = f.
\]

2.5.1. Normal distribution. \(\hat{f}(0) = 1\) gives

\[
\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} dx = 1,
\]

(7)

(one may also use integration on \(\mathbb{R}^2\) to compute the following integral directly, see page 138 formula (6) in [3]), consider

\[
u = \sqrt{tx} + \mu, \quad t > 0, \quad \mu \in \mathbb{R},
\]

then (7) becomes the following classical formula in Gauss’s normal distribution theory

\[
\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi t}} e^{-\frac{(u-\mu)^2}{2t}} du = 1,
\]

(8)

where

\[
f(u | \mu, t) := \frac{1}{\sqrt{2\pi t}} e^{-\frac{(u-\mu)^2}{2t}},
\]

(9)

is the probability density of the normal distribution with expectation \(\mu\) and variance \(t\).
2.6. Fourier transform of derivative and convolution. In this section, we only consider functions that are smooth and rapidly decreasing. The main result is the following:

**Theorem 2.8.** Let $\mathcal{F}(f)(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{-iwx}dx$ be the Fourier transform of $f$. Then

1) $\mathcal{F}(af + bg) = a\mathcal{F}(f) + b\mathcal{F}(g)$;
2) $\mathcal{F}(f') = iw\mathcal{F}(f)$;
3) $(\mathcal{F}(f))^\prime = -i\mathcal{F}(xf(x))$.

**Proof.** We only prove 2). Notice that

$$\int_{-\infty}^{\infty} f'(x)e^{-iwx}dx = \int_{-\infty}^{\infty} f''(x)e^{-iwx}dx = \int_{-\infty}^{\infty} df(e^{-iwx}) = \int_{-\infty}^{\infty} d(e^{-iwx}f) - \int_{-\infty}^{\infty} f(x)d(e^{-iwx}).$$

Since $f$ is smooth and rapidly decreasing, we have

$$\int_{-\infty}^{\infty} d(e^{-iwx}f) = \lim_{x \to \infty} e^{-iwx}f(x) - \lim_{x \to -\infty} e^{-iwx}f(x) = 0.$$

Thus

$$\int_{-\infty}^{\infty} f'(x)e^{-iwx}dx = -\int_{-\infty}^{\infty} f(x) d(e^{-iwx}) = iw\int_{-\infty}^{\infty} f(x)e^{-iwx}dx,$$

which implies 2). \(\square\)

**Example:** $\mathcal{F}(xe^{-\frac{x^2}{2}}) = -iwe^{-\frac{x^2}{2}}$: By 3), we have

$$\mathcal{F}(xe^{-\frac{x^2}{2}}) = i(\mathcal{F}(e^{-\frac{x^2}{2}}))' = i(e^{-\frac{x^2}{2}})' = -iwe^{-\frac{x^2}{2}}.$$

**Convolution:** Let us first recall the definition of convolution for functions $f, g$ defined on $[0, \infty)$:

$$(f \ast g)(t) := \int_{0}^{t} f(\tau)g(t - \tau)\,d\tau.$$  

Notice that if we extend $f, g$ to functions on $\mathbb{R}$ such that

$$f = g = 0, \text{ when } x \leq 0.$$  

Then we can write

$$(f \ast g)(x) := \int_{-\infty}^{\infty} f(u)g(x - u)\,du.$$

**Definition 2.9.** The convolution of two functions on $\mathbb{R}$ is defined by

$$(f \ast g)(x) := \int_{-\infty}^{\infty} f(u)g(x - u)\,du.$$  

Similar as the Laplace transform, we have

(10)  

$$\mathcal{F}(f \ast g) = \sqrt{2\pi} \mathcal{F}(f) \cdot \mathcal{F}(g).$$  

**Proof.** [Need integration on $\mathbb{R}^2$]. We have

$$\sqrt{2\pi} \mathcal{F}(f \ast g) = \int_{-\infty}^{\infty} (f \ast g)(x)e^{-iwx}dx = \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(u)g(x - u)\,du\right)e^{-iwx}dx.$$

Change the order of integration, we get

$$\sqrt{2\pi} \mathcal{F}(f \ast g) = \int_{-\infty}^{\infty} f(u) \left(\int_{-\infty}^{\infty} g(x - u)e^{-iwx}dx\right)\,du = \sqrt{2\pi} \int_{-\infty}^{\infty} f(u)\mathcal{F}(g)(w)e^{-iwx}du,$$

which gives $\sqrt{2\pi} \mathcal{F}(f \ast g) = 2\pi \mathcal{F}(f) \cdot \mathcal{F}(g).$ \(\square\)
3. Partial differential equations

3.1. Functions of several variables. In Laplace transform and Fourier transform, the functions $e^{-st}, e^{-iwx}$ depend on two variables. We can look at $e^{-st}$ as a map, say

$$f : (s, t) \mapsto f(s, t) := e^{-st},$$

from $\mathbb{R}^2$ to $\mathbb{R}$. We say that the map $f$ defines a function on $\mathbb{R}^2$. It is clear that

$$f(x, y) = x^2 + y^2,$$

is a function on $\mathbb{R}^2$;

$$f(x, y, z) = x + y + z,$$

is a function on $\mathbb{R}^3$ and

$$f(t, x) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}},$$

is a function on $(0, \infty) \times \mathbb{R}$.

3.1.1. Graph. If $f$ is a function on $U \subseteq \mathbb{R}^n$ then we call the following set

$$G_f, U := \{(x, f(x)) \in \mathbb{R}^{n+1} : x \in U\},$$

in $\mathbb{R}^{n+1}$ the graph of $f$ over $U$.

**Exercise:** Draw the graph of $f(x, y) = x^2 + y^2$ over the unit disc.

3.1.2. Partial derivatives. $x$-partial derivative of $f(x, y)$ means derivative of $f$ with $y$ fixed:

$$\frac{\partial f}{\partial x}(x, y) := \lim_{h \to 0} \frac{f(x + h, y) - f(x, y)}{h},$$

we write

$$f_x := \frac{\partial f}{\partial x}, \quad f_{xy} := \frac{\partial f_x}{\partial y}.$$

**Example:** If $f(x, y) = x^2 + y^2 + xy$ then

$$f_x = 2x + y, \quad f_y = 2y + x, \quad f_{xy} = 1 = f_{yx}, \quad f_{xx} = 2, \quad f_{yy} = 2.$$

**Example:** If

$$f(t, x) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}},$$

then

$$f_t = \frac{1}{\sqrt{2\pi t}} \cdot \frac{-1}{2} \cdot t^{-\frac{3}{2}} \cdot e^{-\frac{x^2}{2t}} + \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} \cdot \frac{-x^2}{2} \cdot \frac{-1}{t^2} = \frac{x^2 - t}{2t^2} f,$$

and

$$f_x = \frac{x}{t} f, \quad f_{xx} = -\frac{1}{t} f + \frac{x^2}{t^2} f = \frac{x^2 - t}{t^2} f.$$

Thus we get that

(11) $$f_t = \frac{1}{2} f_{xx}.$$
3.1.3. **Directional derivative and gradient.** Let \( n \) in the unit sphere be a given direction, then we call
\[
f_n(p) := \lim_{h \to 0} \frac{f(p + hn) - f(p)}{h},
\]
the derivative of \( f \) along direction \( n \) at \( p \). If we write \( n = (a, b, c) \), \( p = (x, y, z) \), then
\[
f_n(p) = \lim_{h \to 0} \frac{f(x + ah, y + bh, z + ch) - f(x, y, z)}{h}.
\]

**Example:** If
\[
f(x, y, z) = c_0 + c_1x + c_2y + c_3z,
\]
then
\[
f(hn) = f(ah, bh, ch) = c_0 + (c_1a + c_2b + c_3c)h,
\]
which gives
\[
f_n(0) = c_1a + c_2b + c_3c = (f_x(0), f_y(0), f_z(0)) \cdot n.
\]

**Definition 3.1.** We call
\[
\nabla f(p) := (f_{x_1}(p), \cdots, f_{x_m}(p)),
\]
the gradient of \( f(x_1, \cdots, x_m) \) at \( p \).

For a general smooth function, we have the following generalization of (12):
\[
f_n(p) = \nabla f(p) \cdot n.
\]
Since \(|n| = 1\), the above formula gives
\[
f_n(p) = \nabla f(p) \cos \theta,
\]
where \( \theta \) denotes the angle from \( \nabla f(p) \) to \( n \).

3.1.4. **PDEs.** We shall study the following partial differential equations

1. **Wave equation**
\[
u_{tt} = u_{xx}.
\]

2. **Heat equation**
\[
u_t = \frac{1}{2} u_{xx},
\]

**Homework:** Find the background of the above two equations in [2] or wikipedia.

**Example:** By (11), we know that
\[
f(t, x) := \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}},
\]
satisfies the above heat equation. We call \( f(t, x) \) the *heat kernel.*
3.2. Solve wave equation by Fourier series. Let us solve the wave equation

\[ u_{tt} = u_{xx}, \]

with boundary conditions

\[ u(t, 0) = u(t, \pi) = 0, \quad \forall \ t \geq 0; \]

and initial conditions

\[ u(0, x) = f(x), \ u_t(0, x) = g(x), \ \forall \ 0 \leq x \leq \pi. \]

**Step 1: Separating variables:** Find solutions of the form

\[ u(t, x) = G(t)F(x), \]

Since

\[ u_{tt} = G''F, \ u_{xx} = GF'', \]

our equation becomes

\[ G''F = GF'', \]

thus

\[ \frac{G''}{G} = \frac{F''}{F} \equiv k, \]

where \( k \) is constant (notice that \( k \) does not depend on \( t \) and \( x \)).

**Step 2: Fit boundary conditions:** Notice that the boundary conditions

\[ G(t)F(0) = G(t)F(\pi) = 0, \]

is equivalent to

\[ F(0) = F(\pi) = 0. \]

In case \( k = 0 \), then \( F'' \equiv 0 \), i.e \( F(x) = ax + b \). The boundary conditions give \( F \equiv 0 \).

In case \( k = \mu^2 > 0 \), then the general solution for

\[ F'' = \mu^2 F, \]

is \( F = Ae^{\mu x} + Be^{-\mu x} \), then the boundary conditions give

\[ A + B = 0, \ Ae^{\mu \pi} + Be^{-\mu \pi} = 0, \]

thus \( A = B = 0 \).

Thus the only possible case is \( k = -p^2 < 0 \), then the general solution for

\[ F'' = -p^2 F, \]

is \( F = A \cos px + B \sin px, F(0) = 0 \) gives \( A = 0 \). Thus \( F = B \sin px, B \neq 0 \), but \( F(\pi) = 0 \) gives \( \sin p\pi = 0 \), i.e.

\[ p = n, \ n = 1, 2, \ldots, \]

(notice that \( \sin -px = -\sin px \), thus up to a constant they give the same solution).

**Summary:** The boundary condition implies that \( p = n^2, n = 1, 2, \ldots \), and

\[ F = F_n(x) = \sin nx. \]

Now let us solve

\[ G'' = -n^2 G, \]

the general solution is

\[ G_n(t) = B_n \cos nt + C_n \sin nt. \]
Now we know that each
\[ u_n(t, x) = G_n(t) F_n(t) = (B_n \cos nt + C_n \sin nt) \sin nx, \]
satisfies the wave equation and the boundary conditions, so does
\[ u(t, x) = \sum_{n=1}^{\infty} (B_n \cos nt + C_n \sin nt) \sin nx. \]

**Step 3: Fit the initial conditions:** Choose \( B_n \) and \( C_n \) such that
\[ u(0, x) = f(x), \quad u_t(0, x) = g(x). \]

i.e.
\[ \sum_{n=1}^{\infty} B_n \sin nx = f(x); \]

and
\[ \sum_{n=1}^{\infty} nC_n \sin nx = g(x); \]

Consider odd extension \( f_o, g_o \) of \( f, g \), by Theorem 2.6, if \( f_o \) and \( g_o \) is smooth enough, then it is enough to choose
\[ B_n = \frac{2}{\pi} \int_{0}^{\pi} f(x) \sin nx \, dx, \]

and
\[ nC_n = \frac{2}{\pi} \int_{0}^{\pi} g(x) \sin nx \, dx. \]

Now we know that if \( f, g \) have smooth enough odd extension then
\[ u(t, x) = \sum_{n=1}^{\infty} (B_n \cos nt + C_n \sin nt) \sin nx, \]
solves the wave equation (of course we have it check that it converges).

**Example:** When \( g = 0 \) we have
\[ u(t, x) = \sum_{n=1}^{\infty} B_n \cos nt \sin nx. \]

Thus we can write
\[ u(t, x) = \frac{1}{2} \sum_{n=1}^{\infty} (B_n \sin n(x - t) + B_n \sin n(x + t)) = \frac{1}{2} (f_o(x - t) + f_o(x + t)), \]
i.e. \( u(t, x) \) is the superposition of two travelings of the initial wave.

**Exercise:** In case
\[ f(x) = x, \quad 0 \leq x \leq \frac{\pi}{2}, \quad f(x) = \pi - x, \quad \frac{\pi}{2} < x \leq \pi, \]
try to draw the graph of \( u(t, x) \) for \( t = 0, \frac{\pi}{8}, \frac{\pi}{4}, \frac{\pi}{2}, \pi \).
3.3. **Solve heat equation by Fourier series.** Consider the heat equation

\[ u_t = \frac{1}{2} u_{xx}, \]

with boundary conditions

\[ u(t, 0) = u(t, \pi) = 0, \ \forall \ t \geq 0; \]

and initial conditions

\[ u(0, x) = f(x), \ \forall \ 0 \leq x \leq \pi. \]

**Step 1: Separating variables:** Find solutions of the form

\[ u(t, x) = G(t)F(x), \]

Since

\[ u_t = G'F, \quad u_{xx} = GF'', \]

our equation becomes

\[ G'F = \frac{1}{2} GF'', \]

thus

\[ \frac{2G'}{G} = \frac{F''}{F} \equiv k, \]

**Step 2: Fit the boundary conditions:** Same as the wave equation, we have

\[ k = -n^2, \quad n = 1, 2 \cdots, n \]

and

\[ F_n(x) = \sin nx. \]

Then

\[ G' = \frac{n^2}{2} G, \]

gives

\[ G_n(t) = B_n e^{-\frac{n^2}{2} t}, \]

Thus the general solution is

\[ u(t, x) = \sum_{n=1}^{\infty} B_n e^{-\frac{n^2}{2} t} \sin nx. \]

**Step 3: Fit the initial conditions:** We hope

\[ u(0, x) = \sum_{n=1}^{\infty} B_n \sin nx = f(x). \]

By Theorem 2.6, we get

\[ B_n = \frac{2}{\pi} \int_{0}^{\pi} f(x) \sin nx \, dx, \]

**Example:** \( f(x) = 2 \sin x \) then

\[ B_1 = 2, \quad B_2 = \cdots = B_n = 0. \]

Thus

\[ u(t, x) = 2e^{-\frac{t^2}{2}} \sin x, \]

Notice that \( u \) goes to zero as \( t \) goes to infinity.
Exercise: Find $u$ with
\[ f(x) = x, \quad 0 \leq x \leq \frac{\pi}{2}; \quad f(x) = \pi - x, \quad \frac{\pi}{2} \leq x \leq \pi. \]

3.4. Solve heat equation by Fourier transform. Still the heat equation
\[ u_t = \frac{1}{2} u_{xx}. \]
But this time we consider the initial condition for all $x$ in $\mathbb{R}$ (thus no boundary conditions):
\[ u(0, x) = f(x), \quad \forall -\infty < x < \infty. \]

Step 1: Reduce to ODE by Fourier transform: Consider Fourier transform of $u_t$ with respect to the $x$ variable
\[ \mathcal{F}(u_t) = \hat{u}_t(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u_t(t, x)e^{-ixw} \, dx. \]
Then we have
\[ \mathcal{F}(u_t) = \mathcal{F}\left(\frac{1}{2} u_{xx}\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{2} u_{xx}(t, x)e^{-ixw} \, dx. \]
Recall that if $u_x$ is smooth and rapidly decreasing then
\[ \int_{-\infty}^{\infty} u_{xx} e^{-ixw} \, dx = \int_{-\infty}^{\infty} e^{-ixw} d(u_x) = -\int_{-\infty}^{\infty} u_x d(e^{-ixw}) = iw \int_{-\infty}^{\infty} u_x e^{-ixw} \, dx, \]
the same computation for $u$ gives
\[ \int_{-\infty}^{\infty} u_x e^{-ixw} \, dx = iw \int_{-\infty}^{\infty} u e^{-ixw} \, dx, \]
thus we have
\[ \mathcal{F}(u_t) = -\frac{w^2}{2} \mathcal{F}(u), \quad \mathcal{F}(u) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(t, x)e^{-ixw} \, dx\]
Notice that we also have
\[ \mathcal{F}(u_t) = (\mathcal{F}(u))_t. \]
Thus $\mathcal{F}(u)$ satisfies the following ODE.
\[ (\mathcal{F}(u))_t = -\frac{w^2}{2} \mathcal{F}(u). \]

Step 2: Solve the ODE and fit the initial condition: The general solution is
\[ \mathcal{F}(u)(t, w) = c(w)e^{-\frac{w^2}{2} t}. \]
Notice that our initial condition implies
\[ \mathcal{F}(u)(0, w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{-ixw} \, dx = \mathcal{F}(f). \]
Thus
\[ c(w) = \mathcal{F}(f). \]
Now we have
\[ \mathcal{F}(u)(t, w) = \mathcal{F}(f) \cdot e^{-\frac{w^2}{2} t}. \]

Step 3: use Fourier convolution formula: Recall that (see (6))
\[ e^\frac{-y^2}{2} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^\frac{-y^2}{2} e^{-iyu} \, dy. \]
Take
\[ u = w\sqrt{t}, \; y = \frac{x}{\sqrt{t}}, \]
we get
\[ e^{-\frac{w^2}{2}t} = \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2t}} e^{-ixw} dy = \frac{1}{\sqrt{t}} F(e^{-\frac{x^2}{2t}}). \]
Thus
\[ F(u)(t, w) = F(f) \cdot \left( \frac{1}{\sqrt{t}} F(e^{-\frac{x^2}{2t}}) \right) \]
Now the Fourier convolution formula gives
\[ u(t, x) = \frac{1}{\sqrt{2\pi t}} (f \ast e^{-\frac{x^2}{2t}}) = \int_{-\infty}^{\infty} f(p) \frac{1}{\sqrt{2\pi t}} e^{-\frac{(x-p)^2}{2t}} dp, \]

**Summary:** the solution \( u(t, x) \) is given by convolution of the initial temperature distribution with the heat kernel.

4. Appendix: Definition of \( e, \pi \) and Euler’s Formula

4.1. Where does \( e \) come from? Recall that: Let \( A : \mathbb{C}^n \to \mathbb{C}^n \) be a linear map (here linear map means \( A(au + bv) = aA(u) + bA(v) \) for all \( a, b \in \mathbb{C} \) and all \( u, v \in \mathbb{C}^n \)). We call \( u \neq 0 \) in \( \mathbb{C}^n \) an eigenvector of \( A \) if
\[
Au = \lambda u,
\]
where \( \lambda \) is a constant in \( \mathbb{C} \).

**What is an eigenvector of the derivative?**

By (14), we want to find function \( u : \mathbb{R} \to \mathbb{C} \) such that
\[
u' = \lambda u.
\]

**Power series method:** Assume that
\[ u(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots. \]
The following lemma gives:
\[
u'(x) = a_1 + 2a_2 x + \cdots + n a_n x^{n-1} + (n+1) a_{n+1} x^n + \cdots. \]

**Lemma 4.1.** \((x^n)' = nx^{n-1}, \; n = 1, 2, \cdots.\)

**Proof.** If \( n = 1 \) then
\[
x'(x) = \lim_{\Delta x \to 0} \frac{(x + \Delta x) - x}{\Delta x} = 1.
\]
Assume the Lemma for \( n = 1, \cdots, N - 1. \) Then \((fg)' = f'g + fg'\) gives
\[
(x^N)' = (x^{N-1})' \cdot x + x^{N-1} \cdot x' = (N-1)x^{N-2} \cdot x + x^{N-1} = Nx^{N-1}.
\]
The proof is complete.
Exercise: Why we have \((fg)' = f'g + fg'\)?

Now

\[ u' = \lambda u \iff \lambda a_n = (n+1)a_{n+1}, \quad n = 0, 1, \ldots. \]

Thus

\[ a_{n+1} = \frac{\lambda a_n}{n+1} = \frac{\lambda^2 a_{n-1}}{(n+1)n} = \cdots = \frac{\lambda^{n+1} a_0}{(n+1)n \cdots 1} = \frac{\lambda^{n+1} a_0}{(n+1)!}, \]

where we define

\[ n! = 1 \cdot 2 \cdots n. \]

Then we have

\[ u(x) = u_0 \cdot (1 + \lambda x + \cdots + \frac{(\lambda x)^n}{n!} + \cdots). \]

Put

\[ E(x) := 1 + x + \cdots + \frac{x^n}{n!} + \cdots. \]

Since for every \(C > 0\),

\[ \lim_{n \to \infty} \frac{C^n}{n!} = 0, \]

we know that \(E(x)\) converges for all \(x \in \mathbb{C}\).

**Theorem 4.2.** \(E(\lambda x)\) is a unique solution of the eigenvalue equation

\[ u' = \lambda u, \]

with initial condition \(u(0) = 1\).

**Definition 4.3.** We shall define

\[ e := E(1) = 1 + 1 + \frac{1}{2} + \cdots + \frac{1}{n!} + \cdots. \]

4.2. **Definition of the exponential function.** Let us write

\[ e^2 = e \cdot e, \quad e^3 = e^2 \cdot e, \]

and define \(e^m\) inductively by

\[ e^{n+1} = e^n \cdot e. \]

Since \(e\) is positive, we can take the \(q - t\)h root of \(e^m\), we write it as \(e^{\frac{m}{n}}\). Thus for every \(x \in \mathbb{Q}\), \(e^x\) is well defined. The following lemma tells us that \(E(x)\) is an extension of \(e^x\) from \(\mathbb{Q}\) to \(\mathbb{C}\).

**Lemma 4.4.** For every \(x \in \mathbb{Q}\), we have \(e^x = E(x)\).

**Proof.** Since \(E(1) = e\), it suffices to prove

(15) \[ E(\lambda_1)E(\lambda_2) = E(\lambda_1 + \lambda_2), \]

for every \(\lambda_1, \lambda_2 \in \mathbb{C}\). Notice that

\[ (E(\lambda_1 x)E(\lambda_2 x))' = E(\lambda_1 x)'E(\lambda_2 x) + E(\lambda_2 x)'E(\lambda_1 x). \]

Put

\[ G(x) = E(\lambda_1 x)E(\lambda_2 x). \]

Apply \(E(\lambda x)' = \lambda E(\lambda x)\), we get

\[ G' = (\lambda_1 + \lambda_2)G. \]

Notice that \(G(0) = 1\). Thus Theorem 4.2 implies that

\[ G(x) = E((\lambda_1 + \lambda_2)x). \]
Take $x = 1$, we get $E(\lambda_1)E(\lambda_2) = E(\lambda_1 + \lambda_2)$. □

**Exercise:** Find a direct proof of $E(\lambda_1)E(\lambda_2) = E(\lambda_1 + \lambda_2)$ without using Theorem 4.2.

**Definition 4.5.** We shall use the same symbol $e^x$ to denote $E(x)$ for all $x$ in $\mathbb{C}$ and call $e^x$ the exponential function. If $x > 0$ then we define $\ln x$ as the unique real solution of $e^{\ln x} = x$.

By Theorem 4.2, we know that $e^x$ is fully determined by

$$(e^x)' = e^x, \quad e^0 = 1.$$ 

4.3. **Definition of $\pi$ and trigonometric functions.** : Fix $P_0 = (1, 0)$ in the unit circle $S^1 := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$.

A counterclockwise rotation of $P_0$ gives a arc $P_0P$. The length, say $\theta(P)$, of the arc $P_0P$ is a function of $P$. It is clear that the circumference diameter ratio is equal to $\theta(-1, 0)$.

**Definition 4.6 (Definition of $\pi$).** We shall write the circumference diameter ratio as $\pi$.

Denote by $F : \theta(P) \mapsto P$,

the inverse function of $0 \leq \theta(P) \leq 2\pi$.

**Definition 4.7.** We shall write $F(\theta) = (\cos \theta, \sin \theta)$.

Notice that

$$F(0) = (1, 0) = F(2\pi), \quad F(\pi) = (-1, 0), \quad |F(\theta)| \equiv 1.$$ 

In particular, it gives

$$\sin(0) = \sin(2\pi) = 0, \quad \cos(0) = \cos(2\pi) = 1.$$ 

By definition of $\theta$, we have

$$\int_0^\theta |F'(\theta)| \, d\theta = \dot{\theta}, \quad 0 \leq \dot{\theta} \leq 2\pi,$$

which gives

$$|F'(\theta)| \equiv 1.$$ 

Now $F(\theta) \cdot F'(\theta) \equiv 1$ implies

$$F' \cdot F + F \cdot F' = 2F \cdot F' \equiv 0.$$ 

Hence $F' \perp F$, thus we know that

$$F'(\theta) = (- \sin \theta, \cos \theta), \quad \text{or} \quad F'(\theta) = (\sin \theta, - \cos \theta).$$ 

But notice that $F'(0) = (0, 1)$, thus we must have

$$F'(\theta) = (- \sin \theta, \cos \theta),$$ 

which is equivalent to

$$\left(\cos \theta + i \sin \theta\right)' = i(\cos \theta + i \sin \theta).$$ 

Notice that $\cos 0 + i \sin 0 = 1$, thus Theorem 4.2 gives

**Theorem 4.8 (Euler’s formula).** $e^{i\theta} = \cos \theta + i \sin \theta$. 

Take $\theta = \pi$, we get the following Euler’s identity
\[ e^{i\pi} = -1. \]
Moreover, apply (15), we get
\[ e^{i\theta_1}e^{i\theta_2} = e^{i(\theta_1+\theta_2)}, \]
thus by Euler’s formula, we have
\[ (\cos \theta_1 + i \sin \theta_1)(\cos \theta_2 + i \sin \theta_2) = \cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2), \]
\[ \text{i.e.} \]
\[ \cos(\theta_1 + \theta_2) = \cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2, \]
and
\[ \sin(\theta_1 + \theta_2) = \sin \theta_1 \cos \theta_2 + \cos \theta_1 \sin \theta_2. \]

REFERENCES

DEPARTMENT OF MATHEMATICAL SCIENCES, NORWEGIAN UNIVERSITY OF SCIENCE AND TECHNOLOGY, NO-7491 TRONDHEIM, NORWAY
E-mail address: xu.wang@ntnu.no