1. Introduction

These are lecture notes on *Laplace transform, Fourier transform and their applications* by Xu Wang based on Erwin Kreyszig's book *Advanced engineering mathematics* (10 th edition), Dag Wessel-Berg's video: *http://video.adm.ntnu.no/serier/4fe2d4d3dbe03* and references [1, 3, 4].

Contents

2. What is e ?

Eigenvector: Recall that if $A : \mathbb{C}^n \to \mathbb{C}^n$ is linear, we call $u \neq 0$ in $\mathbb{C}^n$ an *eigenvector* of A if

$$Au = \lambda u, \tag{1}$$

where λ is a constant in $\mathbb{C}$.

Eigenvector of the derivative: In this course, we will answer the following question first:

What is an eigenvector of the derivative ?

By (1), we want to find function $u : \mathbb{R} \to \mathbb{C}$ such that

$$u' = \lambda u.$$

Assume that

$$u(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots .$$

The following lemma gives:

$$u'(x) = a_1 + 2a_2 x + \cdots + n a_n x^{n-1} + (n+1) a_{n+1} x^n + \cdots .$$

Lemma 2.1. $(x^n)' = n x^{n-1},\ n = 1, 2, \cdots .$

Proof. If $n = 1$ then

$$x'(x) = \lim_{\triangle x \to 0} \frac{(x + \triangle x) - x}{\triangle x} = 1.$$

Assume the Lemma for $n = 1, \cdots, N-1$. Then $(fg)' = f'g + fg'$ gives

$$(x^N)' = (x^{N-1})' \cdot x + x^{N-1} \cdot x' = (N-1) x^{N-2} \cdot x + x^{N-1} = N x^{N-1}.$$

The proof is complete. □

Exercise: Why we have $(fg)' = f'g + fg'$?

Now

$$u' = \lambda u \Leftrightarrow \lambda a_n = (n+1) a_{n+1},\ n = 0, 1, \cdots .$$

Thus

$$a_{n+1} = \frac{\lambda a_n}{(n+1)} = \frac{\lambda^2 a_{n-1}}{(n+1)n} = \cdots = \frac{\lambda^{n+1} a_0}{(n+1)n \cdots 1} = \frac{\lambda^{n+1} a_0}{(n+1)!},$$

where we define

$$n! = 1 \cdot 2 \cdots n.$$

Then we have

$$u(x) = u_0 \cdot (1 + \lambda x + \cdots + \frac{(\lambda x)^n}{n!} + \cdots).$$

Put

$$E(x) := 1 + x + \cdots + \frac{x^n}{n!} + \cdots .$$

Since for every $C > 0$,

$$\lim_{n \to \infty} \frac{C^n}{n!} = 0,$$

we know that $E(x)$ converges for all $x \in \mathbb{C}$.

Theorem 2.2. *$E(\lambda x)$ is a unique solution of the eigenvalue equation*

$$u' = \lambda u,$$

with initail condition $u(0) = 1$.

Definition 2.3. *We shall define*

$$e := E(1) = 1 + 1 + \frac{1}{2} + \cdots + \frac{1}{n!} + \cdots .$$

3. Exponential function

Let us write

$$e^2 = e \cdot e,\ e^3 = e^2 \cdot e,$$

and define e^m inductively by

$$e^{n+1} = e^n \cdot e.$$

Since e is positive, we can take the $q-th$ roof of e^m, we write it as $e^{\frac{m}{q}}$. Thus for every $x \in \mathbb{Q}$, e^x is well defined. The following lemma tells us that $E(x)$ is an extension of e^x from $\mathbb{Q}$ to $\mathbb{C}$.

Lemma 3.1. *For every $x \in \mathbb{Q}$, we have $e^x = E(x)$.*

Proof. Since $E(1) = e$, it suffices to prove

$$E(\lambda_1)E(\lambda_2) = E(\lambda_1 + \lambda_2), \tag{2}$$

for every λ_1, λ_2 in $\mathbb{C}$. Notice that

$$(E(\lambda_1 x)E(\lambda_2 x))' = E(\lambda_1 x)'E(\lambda_2 x) + E(\lambda_2 x)'E(\lambda_1 x).$$

Put

$$G(x) = E(\lambda_1 x)E(\lambda_2 x).$$

Apply $E(\lambda x)' = \lambda E(\lambda x)$, we get

$$G' = (\lambda_1 + \lambda_2)G.$$

Notice that $G(0) = 1$. Thus Theorem 2.2 implies that

$$G(x) = E((\lambda_1 + \lambda_2)x).$$

Take $x = 1$, we get $E(\lambda_1)E(\lambda_2) = E(\lambda_1 + \lambda_2)$. □

Exercise: Find a direct proof of $E(\lambda_1)E(\lambda_2) = E(\lambda_1 + \lambda_2)$ without using Theorem 2.2.

Definition 3.2. *We shall use the same symbol e^x to denote $E(x)$ for all x in $\mathbb{C}$ and call e^x the* ***exponential function****. If $x > 0$ then we define $\ln x$ as the unique real solution of $e^{\ln x} = x$.*

By Theorem 2.2, we know that e^x is fully determined by

$$(e^x)' = e^x,\ e^0 = 1.$$

Jordan normal form of the derivative (this part is not assumed in the course) In linear algebra, we know that if $A : \mathbb{C}^n \to \mathbb{C}^n$ is not diagonalizable then we need to find u such that

$$(A - \lambda)^m u = 0,$$

for some positive integer m. In our case, if $\lambda = 0$ then

$$\frac{d^m}{dx^m} u = 0,$$

if and only if u is polynomial of degree $m - 1$. In general, one may check that

$$(\frac{d}{dx} - \lambda)^m u = 0,$$

if and only if $u(x)e^{-\lambda x}$ is a polynomial of degree $m - 1$. In linear algebra, we hope to write

$$u = a_1 u_1 + \cdots + a_k u_k,$$

where u_k satisfies

$$(A - \lambda_k)^{m_k} u_k = 0.$$

In case A is the derivative, then it suggests to write a function u as a

$$u(x) = \int e^{\lambda x} \, a_0(\lambda) + x \cdot \int e^{\lambda x} \, a_1(\lambda) + \cdots + x^k \cdot \int e^{\lambda x} \, a_k(\lambda) + \cdots,$$

where each $a_j(\lambda)$ is a measure in $\mathbb{C}$. If we only consider real λ then we call

$$F(s) = \int_0^\infty e^{-st} f(t) \, dt,$$

the *Laplace transform* of f. In general, we call

$$\hat{f}(w) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-iwv} f(v) \, dv, \; i := \sqrt{-1},$$

the *Fourier transform* of f.

Recall definition of π and trigonometric functions: Fix $P_0 = (1, 0)$ in the unit circle

$$S^1 := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}.$$

A counterclockwise rotation of P_0 gives a arc P_0P. The length, say $\theta(P)$, of the arc P_0P is a function of P. It is clear that the circumference diameter ratio is equal to $\theta(-1, 0)$.

Definition 3.3 (Definition of π)**.** *We shall write the circumference diameter ratio as* π.

Denote by

$$F : \theta(P) \mapsto P,$$

the inverse function of $0 \leq \theta(P) \leq 2\pi$.

Definition 3.4. *We shall write* $F(\theta) = (\cos\theta, \sin\theta)$.

Notice that

$$F(0) = (1, 0) = F(2\pi), \; F(\pi) = (-1, 0), \; |F(\theta)| \equiv 1.$$

In particular, it gives

$$\sin(0) = \sin(2\pi) = 0, \; \cos(0) = \cos(2\pi) = 1.$$

By definition of θ, we have

$$\int_0^{\hat{\theta}} |F'(\theta)| \, d\theta = \hat{\theta}, \; 0 \leq \hat{\theta} \leq 2\pi,$$

which gives

$$|F'(\theta)| \equiv 1.$$

Now $F(\theta) \cdot F(\theta) \equiv 1$ implies

$$F' \cdot F + F \cdot F' = 2F \cdot F' \equiv 0.$$

Hence $F' \perp F$, thus we know that

$$F'(\theta) = (-\sin\theta, \cos\theta), \text{ or } F'(\theta) = (\sin\theta, -\cos\theta).$$

But notice that $F'(0) = (0, 1)$, thus we must have

$$F'(\theta) = (-\sin\theta, \cos\theta),$$

which is equivalent to

$$(\cos\theta + i\sin\theta)' = i(\cos\theta + i\sin\theta).$$

Notice that $\cos 0 + i \sin 0 = 1$, thus Theorem 2.2 gives

Theorem 3.5 (Euler's formula)**.** $e^{i\theta} = \cos\theta + i\sin\theta$.

Take $\theta = \pi$, we get the following Euler's identity

$$e^{i\pi} = -1.$$

Moreover, apply (2), we get

$$e^{i\theta_1}e^{i\theta_2} = e^{i(\theta_1+\theta_2)},$$

thus by Euler's formula, we have

$$(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) = \cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2),$$

i.e.

$$\cos(\theta_1+\theta_2) = \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2, \tag{3}$$

and

$$\sin(\theta_1+\theta_2) = \sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2. \tag{4}$$

4. Laplace transform, basic facts

Definition 4.1. *Let $f(t), t \geq 0$ be a given function. We call*

$$F(s) := \int_0^\infty e^{-st} f(t)\, dt,$$

*the **Laplace transform** of $f(t)$. and write*

$$F = \mathcal{L}(f),\ f = \mathcal{L}^{-1}F.$$

In order to compute Laplace transforms, we need the following two *fundamental formulas*:

$$d(fg) = fdg + gdf,\ \int_a^b df = f(b) - f(a), \tag{5}$$

where $df := f'(t)dt$.

Example: $\mathcal{L}(1)$: Consider

$$f(t) = 1,\ t \geq 0.$$

Then

$$F(s) = \int_0^\infty e^{-st}\cdot 1\, dt = \int_0^\infty d(\frac{e^{-st}}{-s}) = \frac{e^{-st}}{-s}|_{t=\infty} - \frac{e^{-st}}{-s}|_{t=0} = 0 - \frac{1}{-s} = \frac{1}{s},\ s > 0.$$

Thus we get

$$\mathcal{L}(1) = \frac{1}{s},\ \ \mathcal{L}^{-1}(\frac{1}{s}) = 1,\ \ s > 0. \tag{6}$$

Example: $\mathcal{L}(e^{kt}),\ k \in \mathbb{R}$: We have

$$\mathcal{L}(e^{kt}) = \int_0^\infty e^{-st}e^{kt}\, dt = \int_0^\infty e^{-(s-k)t}\, dt = \mathcal{L}(1)(s-k) = \frac{1}{s-k},\ \ s > k.$$

Thus

$$\mathcal{L}(e^{kt}) = \frac{1}{s-k},\ \ \mathcal{L}^{-1}(\frac{1}{s-k}) = e^{kt},\ \ s > k. \tag{7}$$

Example: $\mathcal{L}(e^{t^2})$ **does not exist**: Notice that

$$\int_0^\infty e^{-st}e^{t^2}\, dt = \infty,$$

for all real number s.

Remark: Laplace transform is linear: For every pair of real numbers a, b, we have

(8) $$\mathcal{L}(af + bg) = a\mathcal{L}(f) + b\mathcal{L}(g).$$

Example: $\mathcal{L}(3 + 2e^{5t})$: We have

$$\mathcal{L}(3 + 2e^{5t}) = 3\mathcal{L}(1) + 2\mathcal{L}(e^{5t}) = 3 \cdot \frac{1}{s} + 2 \cdot \frac{1}{s-5} = \frac{3}{s} + \frac{2}{s-5} = \frac{5(s-3)}{s-5},\ s > 5.$$

Compute inverse Laplace transform: $\mathcal{L}^{-1}(\frac{1}{s^2-3s+2})$: Notice that

$$\frac{1}{s^2 - 3s + 2} = \frac{1}{(s-1)(s-2)} = \frac{1}{s-2} - \frac{1}{s-1}.$$

Thus

$$\mathcal{L}^{-1}(\frac{1}{s^2 - 3s + 2}) = \mathcal{L}^{-1}(\frac{1}{s-2}) - \mathcal{L}^{-1}(\frac{1}{s-1}) = e^{2t} - e^t.$$

Proposition 4.2. $\mathcal{L}(t^n) = \frac{n!}{s^{n+1}}$, $n = 1, 2, \cdots$, $s > 0$.

Proof. Put

$$F_n(s) = \mathcal{L}(t^n).$$

Then by (5), we have

$$F_n(s) = \int_0^\infty e^{-st} t^n\, dt = \int_0^\infty t^n d(\frac{e^{-st}}{-s}) = \int_0^\infty d(t^n \frac{e^{-st}}{-s}) - \frac{e^{-st}}{-s} d(t^n).$$

Since

$$\int_0^\infty d(t^n \frac{e^{-st}}{-s}) = 0 - 0, \text{ if } s > 0,$$

and

$$\int_0^\infty -\frac{e^{-st}}{-s} d(t^n) = \frac{n}{s} \cdot F_{n-1}(s).$$

We get

$$F_n(s) = \frac{n}{s} \cdot F_{n-1}(s) = \frac{n(n-1)}{s^2} \cdot F_{n-2}(s) = \cdots = \frac{n!}{s^n} \cdot F_0(s) = \frac{n!}{s^n}\mathcal{L}(1) = \frac{n!}{s^{n+1}}.$$

□

Remark: Recall that $(t^n)' = nt^{n-1}$, $n = 1, 2, \cdots$, thus the above proposition gives

$$\mathcal{L}((t^n)') = n\mathcal{L}(t^{n-1}) = \frac{n!}{s^n} = s\mathcal{L}(t^n).$$

In general, we have the following theorem

Theorem 4.3 (Laplace transform for derivative)**.**

$$\mathcal{L}(f') = sF(s) - f(0), \qquad \textit{where} \quad F(s) := \mathcal{L}(f).$$

We will prove the above theorem later, first let us show how to use it to solve first order differential equations:

Verify Theorem 2.2: Consider

$$y' = y, \ y(0) = 1.$$

Put

$$Y(s) = \mathcal{L}(y).$$

Then we have

$$L(y') = L(y) = Y(s),$$

Since

$$L(y') = sY(s) - y(0) = sY(s) - 1,$$

we have

$$sY(s) - 1 = Y(s),$$

which gives

$$Y(s) = \frac{1}{s-1}.$$

Thus

$$y(t) = \mathcal{L}^{-1}(\frac{1}{s-1}) = e^t,$$

which verifies Theorem 2.2 for $\lambda = 1$.

5. Laplace transform of derivatives: how to use it to solve differential equations ?

The *main theorem* in Laplace transform is the following:

Theorem 5.1 (Laplace transform of the derivative)**.**

$$\mathcal{L}(f') = s\mathcal{L}(f) - f(0).$$

Proof. By (5), we have

$$\mathcal{L}(f') = \int_0^\infty e^{-st} f'(t)\, dt = \int_0^\infty e^{-st}\, df = \int_0^\infty d(e^{-st} f) - \int_0^\infty f d(e^{-st}).$$

Assume that for some real number k, we have

$$\lim_{t \to \infty} e^{-kt} f(t) = 0.$$

Then for $s \geq k$, we have

$$\int_0^\infty d(e^{-st} f) = \lim_{t \to \infty} e^{-st} f(t) - f(0) = -f(0).$$

Since

$$-\int_0^\infty f d(e^{-st}) = s \int_0^\infty e^{-st} f(t)\, dt = \mathcal{L}(f),$$

we get $\mathcal{L}(f') = s\mathcal{L}(f) - f(0)$ on $s \geq k$. $\square$

Remark 1: In the above proof, we use an extra assumption: for some $k \in \mathbb{R}$,

$$\lim_{t\to\infty} e^{-kt} f(t) = 0,$$

in this course, we only consider functions that satisfy the above condition.

Remark 2: Apply the theorem to f', we get

$$\mathcal{L}(f'') = sL(f') - f'(0) = s(s\mathcal{L}(f) - f(0)) - f'(0) = s^2\mathcal{L}(f) - sf(0) - f'(0).$$

In general, denote by $f^{(n)}$ the n-th order derivative of f then we have

Theorem 5.2 (Laplace transform of n-th order derivative)**.**

$$\mathcal{L}(f^{(n)}) = s^n\mathcal{L}(f) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0).$$

Remark: How to use Laplace transform of derivatives to solve differential equation: By the above theorem, if we apply the Laplace transform to a differential equation

$$y'' + ay' + by = c(t),\ a'b \in \mathbb{R},$$

then we get

$$s^2Y - sy(0) - y'(0) + a(sY - y(0)) + bY = C.$$

Thus

$$(s^2 + as + b)Y = (s + a)y(0) + y'(0) + C,$$

which gives

$$y = \mathcal{L}^{-1}(Y) = \mathcal{L}^{-1}\left(\frac{(s + a)y(0) + y'(0) + C}{s^2 + as + b}\right).$$

Example: Consider

$$y'' + 4y' + 4y = 0,\ \ y(0) = 0, y'(0) = 1.$$

then the above formula gives

$$y = \mathcal{L}^{-1}\left(\frac{1}{s^2 + 4s + 4}\right) = \mathcal{L}^{-1}\left(\frac{1}{(s + 2)^2}\right).$$

How to compute the inverse Laplace transform of $\frac{1}{(s+2)^2}$? Is it related to $\mathcal{L}^{-1}(\frac{1}{s^2}) = t$? We will introduce a simple method to answer these two questions.

6. s–SHIFTING: REPLACING s BY $s - a$ IN THE TRANSFORM

Notice that

$$\int_0^\infty e^{-st}e^{at}f(t)\,dt = F(s - a),\ F(s) = \mathcal{L}(f) = \int_0^\infty e^{-st}f(t)\,dt.$$

thus we get the following theorem:

Theorem 6.1 (s-Shifting theorem)**.**

$$\mathcal{L}(e^{at}f(t)) = F(s - a),\ \ \mathcal{L}^{-1}(F(s - a)) = e^{at}\mathcal{L}^{-1}(F(s)).$$

Compute $\mathcal{L}^{-1}\left(\frac{1}{(s+2)^2}\right)$: In this case

$$F(s) = \frac{1}{s^2},\ \ F(s-(-2)) = \frac{1}{(s+2)^2}.$$

thus $a=-2$. Since $\mathcal{L}^{-1}(\frac{1}{s^2}) = t$, apply the above theorem, we get

$$\mathcal{L}^{-1}\left(\frac{1}{(s+2)^2}\right) = e^{-2t}\mathcal{L}^{-1}(\frac{1}{s^2}) = e^{-2t}t.$$

Compute $\mathcal{L}(e^{kt})$ **again**: Since $\mathcal{L}(1) = \frac{1}{s}$, we get

$$\mathcal{L}(e^{kt}) = \frac{1}{s-k}.$$

In case $k = iw$, we get

$$\mathcal{L}(e^{iwt}) = \frac{1}{s-iw} = \frac{s+iw}{s^2+w^2}.$$

Recall Euler's formula $e^{iwt} = \cos wt + i\sin wt$, thus

(9) $$\mathcal{L}(\cos wt) = \frac{s}{s^2+w^2},\ \mathcal{L}(\sin wt) = \frac{w}{s^2+w^2}.$$

Compute $\mathcal{L}^{-1}(\frac{s+2}{s^2+4})$: Apply the above formula for $w=2$, we get

$$\mathcal{L}^{-1}(\frac{s+2}{s^2+4}) = \mathcal{L}^{-1}(\frac{s}{s^2+4}) + \mathcal{L}^{-1}(\frac{2}{s^2+4}) = \cos 2t + \sin 2t.$$

Compute $\mathcal{L}^{-1}(\frac{s}{s^2+2s+2})$: Since

$$s^2+2s+2 = (s+1)^2+1,$$

we get

$$\frac{s}{s^2+2s+2} = \frac{s}{(s+1)^2+1} = \frac{s+1-1}{(s+1)^2+1} = \frac{s+1}{(s+1)^2+1} - \frac{1}{(s+1)^2+1}.$$

Thus $a=-1$ and

$$\mathcal{L}^{-1}(\frac{s}{s^2+2s+2}) = e^{-t}\mathcal{L}^{-1}(\frac{s}{s^2+1}) - e^{-t}\mathcal{L}^{-1}(\frac{1}{s^2+1}) = e^{-t}(\cos t - \sin t).$$

7. More Laplace transform formulas

Example: Solve $y''-y=t,\ y(0)=y'(0)=1$: Recall that

$$L(y'') = s^2Y - sy(0) - y'(0) = s^2Y - s - 1.$$

Apply Laplace transform to our equation, we get

$$s^2Y - s - 1 - Y = \mathcal{L}(t) = \frac{1}{s^2}.$$

Thus

$$(s^2-1)Y = s+1+\frac{1}{s^2},$$

i.e

$$Y = \frac{s+1}{s^2-1} + \frac{1}{s^2(s^2-1)} = \frac{1}{s-1} + \frac{1}{s^2(s^2-1)}.$$

Notice that

$$\frac{1}{s^2(s^2-1)} = \frac{1}{s^2-1} - \frac{1}{s^2} = \frac{1}{2}\left(\frac{1}{s-1} - \frac{1}{s+1}\right) - \frac{1}{s^2}.$$

Thus

$$y = \mathcal{L}^{-1}(Y) = e^t + \frac{1}{2}(e^t - e^{-t}) - t.$$

Let us introduce the following definition.

Definition 7.1 ($\sinh t$ and $\cosh t$)**.**

$$\sinh t := \frac{e^t - e^{-t}}{2}, \quad \cosh t := \frac{e^t + e^{-t}}{2}.$$

Exercise: $\mathcal{L}(\sinh t) = \frac{1}{s^2-1}$, $\mathcal{L}(\cosh t) = \frac{s}{s^2-1}$.

Example: Consider $f(t)$ such that

$$f(t) = 1, \text{ if } 3 < t < 4; \quad f(t) = 0, \text{ if } 0 \leq t \leq 3 \text{ or } t \geq 4.$$

Then

$$\mathcal{L}(f) = \int_0^\infty f(t)e^{-st}\,dt = \int_3^4 1 \cdot e^{-st}\,dt = \int_3^4 d(\frac{e^{-st}}{-s}) = \frac{e^{-4s}}{-s} - \frac{e^{-3s}}{-s}.$$

Thus

$$\mathcal{L}(f) = \frac{e^{-3s} - e^{-4s}}{s}.$$

Laplace transform of integrals: Put

$$g(t) = \int_0^t f(\tau)\,d\tau,$$

then

$$g' = f, \; g(0) = 0.$$

Thus

$$F = \mathcal{L}(f) = \mathcal{L}(g') = sG - g(0) = sG,$$

which gives $G = \frac{F}{s}$, i.e.

$$\mathcal{L}\left(\int_0^t f(\tau)\,d\tau\right) = \frac{\mathcal{L}(f)}{s}, \quad \int_0^t f(\tau)\,d\tau = \mathcal{L}^{-1}\left(\frac{\mathcal{L}(f)}{s}\right).$$

Compute $\mathcal{L}^{-1}\left(\frac{1}{s(s^2+1)}\right)$: Since

$$\mathcal{L}(\sin t) = \frac{1}{s^2+1},$$

we get

$$\mathcal{L}^{-1}\left(\frac{1}{s(s^2+1)}\right) = \mathcal{L}^{-1}(\frac{\mathcal{L}(\sin t)}{s}) = \int_0^t \sin\tau\,d\tau = 1 - \cos t.$$

Compute $\mathcal{L}^{-1}\left(\frac{1}{s^2(s-1)}\right)$: we have

$$\mathcal{L}^{-1}\left(\frac{1}{s} \cdot \frac{1}{(s-1)}\right) = \int_0^t e^\tau\,d\tau = e^t - 1.$$

thus

$$\mathcal{L}^{-1}\left(\frac{1}{s}\cdot\frac{1}{s(s-1)}\right)=\int_0^t (e^\tau-1)\,d\tau=e^t-1-t.$$

Laplace transform table:

$$\mathcal{L}(f)=\int_0^\infty e^{-st}f(t)\,dt=F(s);$$

$$\mathcal{L}(e^{kt})=\frac{1}{s-k};$$

$$\mathcal{L}(t^n)=\frac{n!}{s^{n+1}};$$

$$\mathcal{L}(\cos wt)=\frac{s}{s^2+w^2},\ \mathcal{L}(\sin wt)=\frac{w}{s^2+w^2};$$

$$\mathcal{L}(f')=sF(s)-f(0);$$

$$\mathcal{L}(f'')=s^2F(s)-sf(0)-f'(0);$$

$$\mathcal{L}(e^{kt}f(t))=F(s-k);$$

$$\mathcal{L}\left(\int_0^t f(\tau)\,d\tau\right)=\frac{F(s)}{s}.$$

Remark: Integration by parts is enough for all the above formulas!

8. Using step functions

Definition 8.1 (Step function)**.** *Let $a\geq 0$, the step function $u(t-a)$ is defined as follows*

$$u(t-a)=0,\ \textit{for}\ 0\leq t<a,\ \ u(t-a)=1,\ \textit{for}\ t\geq a.$$

In case $a=0$ we call $u(t)$ the Heaviside function.

Exercise: Draw the graph of $f(t)=u(t-1)-u(t-3)$.

Exercise: Draw the graph of $f(t)=(u(t)-u(t-\pi))\sin t$.

Exercise: Draw the graph of $f(t)=u(t)+u(t-1)+\cdots+u(t-n)+\cdots$.

Laplace transform of the step function:

$$\mathcal{L}(u(t-a))=\int_0^\infty e^{-st}u(t-a)\,dt=\int_a^\infty e^{-st}\,dt=\int_a^\infty d(\frac{e^{-st}}{-s})=\frac{e^{-as}}{s}.$$

Exercise: Compare the graph of $u(t-a)f(t-a)$ with the graph of $f(t)$.

Theorem 8.2 (t-Shifting theorem)**.**

$$\mathcal{L}(u(t-a)f(t-a))=e^{-as}\mathcal{L}(f).$$

Proof. We have

$$\mathcal{L}(u(t-a)f(t-a)) = \int_a^\infty e^{-st} f(t-a)\ , dt.$$

Consider $t - a = x$, i.e. $t = x + a$, then

$$\int_0^\infty e^{-s(x+a)} f(x)\, d(x+a) = \int_0^\infty e^{-s(x+a)} f(x)\, dx = e^{-as} \int_0^\infty e^{-sx} f(x)\, dx.$$

The right hand side is just $e^{-as}\mathcal{L}(f)$. □

Remark: Compare with the t-Shifting formula

$$\mathcal{L}(e^{at}f(t)) = \mathcal{L}(f)(s-a).$$

Example: $\mathcal{L}^{-1}(e^{-s}\frac{1}{s-2})$ Since

$$\frac{1}{s-2} = \mathcal{L}(e^{2t}),$$

we have

$$\mathcal{L}^{-1}(e^{-s}\frac{1}{s-2}) = \mathcal{L}^{-1}(e^{-s}\mathcal{L}(e^{2t})).$$

Apply inverse Laplace transform to the above theorem, we get

$$\mathcal{L}^{-1}(e^{-s}\mathcal{L}(e^{2t})) = u(t-1)e^{2(t-1)}.$$

Can you draw the graph of $u(t-1)e^{2(t-1)}$?

RC-Circuit equation (see page 29 section 1.5 and page 93 section 2.9 of Kreyszig's book): R, C postive constants, $i(t), e(t)$ functions:

$$Ri(t) + \frac{1}{C}\int_0^t i(\tau)\, d\tau = e(t).$$

Apply the Laplace transform, we get

$$RI(s) + \frac{1}{C}\cdot\frac{I(s)}{s} = E(s),$$

i.e.

$$I(s) = \frac{E(s)}{R + \frac{1}{Cs}} = \frac{s}{s + \frac{1}{RC}}\frac{E(s)}{R}.$$

Example: Consider

$$e(t) = u(t-1) - u(t-2), \ \ R = C = 1.$$

Then

$$E(s) = \frac{e^{-s}}{s} - \frac{e^{-2s}}{s},$$

Thus

$$I(s) = \frac{e^{-s} - e^{-2s}}{s+1}.$$

Since

$$\mathcal{L}(e^{-t}) = \frac{1}{s+1},$$

we get

$$I(s) = e^{-s}\mathcal{L}(e^{-t}) - e^{-2s}\mathcal{L}(e^{-t}).$$

Thus

$$i(s) = u(t-1)e^{-(t-1)} - u(t-2)e^{-(t-2)}.$$

Example: Let

$$f(t) = \sum_{n=0}^{\infty} u(t-n),$$

then

$$\mathcal{L}(f) = \sum_{n=0}^{\infty} \mathcal{L}(u(t-n)) = \sum_{n=0}^{\infty} \frac{e^{-ns}}{s} = \frac{1}{s} \sum_{n=0}^{\infty} (e^{-s})^n,$$

Recall that

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r},$$

thus

$$\mathcal{L}\left(\sum_{n=0}^{\infty} u(t-n)\right) = \frac{1}{s(1-e^{-s})}.$$

9. Dirac delta function

Mathematical definition of the Delta function: $\delta(t-a)$ is a map:

$$\delta(t-a) : f \mapsto f(a).$$

Consider $f(s) = e^{-st}$, then

$$f(a) = e^{-as}.$$

Thus

$$\delta(t-a)(e^{-st}) = e^{-as}.$$

Definition 9.1. *We shall define Laplace transform of* $\delta(t-a)$ *as* $\delta(t-a)(e^{-st})$, *i.e.*

$$\mathcal{L}(\delta(t-a)) = e^{-as}.$$

Intuitive definition of the Delta function: $\delta(t-a)$ is defined by the following "formal" integral:

$$\int_0^{\infty} f(t)\delta(t-a)\, dt = f(a).$$

Consider function

$$d_k(t-a) = \frac{1}{k}, \text{ if } a \leq t \leq a+k; \;\; d_k(t-a) = 0, \text{ otherwise},$$

then

$$\lim_{k\to 0} \int_0^{\infty} f(t) d_k(t-a)\, dt = \lim_{k\to 0} \frac{1}{k} \int_a^{a+k} f(t)\, dt = f(a).$$

Thus we can think of $\delta(t-a)$ as the limit of $d_k(t-a)$, i.e.

$$\int_0^{\infty} f(t)\delta(t-a)\, dt = \lim_{k\to 0} \int_0^{\infty} f(t) d_k(t-a)\, dt = f(a).$$

In case $f(t) = e^{-st}$, the above formula gives

$$\int_0^{\infty} e^{-st}\delta(t-a)\, dt = e^{-as}.$$

That is the reason why we say that e^{-as} is the Laplace transform of $\delta(t-a)$.

Example: Relation with the step function:

$$\int_0^t \delta(\tau - a)\, d\tau = u(t-a).$$

Application: solve

$$y'' + y = \delta(t-1), \ \ y(0) = y'(0) = 0.$$

Apply Laplace transform to the equation, we get

$$s^2 Y + Y = e^{-s}.$$

Thus

$$Y = \frac{e^{-s}}{s^2+1}.$$

Notice that

$$\mathcal{L}(\sin t) = \frac{1}{s^2+1},$$

thus

$$y = \mathcal{L}^{-1}(e^{-s}\mathcal{L}(\sin t)) = u(t-1)\sin(t-1).$$

10. Laplace transform: use known formulas or compute by yourself.

Let us compute Laplace transform of

$$f(t) = t, \text{ if } 0 \le t \le a, \ \ f(t) = 0, \text{ if } t > a.$$

Method 1: using $\mathcal{L}(u(t-a)f(t-a)) = e^{-as}F(s)$. First, let us write

$$f(t) = (1 - u(t-a))t = t - u(t-a)t = t - u(t-a)(t-a) - au(t-a).$$

Thus

$$\mathcal{L}(f) = \mathcal{L}(t) - \mathcal{L}(u(t-a)(t-a)) - a\mathcal{L}(u(t-a)) = \frac{1}{s^2} - \frac{e^{-as}}{s^2} - a\frac{e^{-as}}{s}.$$

Method 2: compute by yourself:

$$\int_0^\infty e^{-st} f(t)\, dt = \int_0^a te^{-st}\, dt = \int_0^a t\, d(\frac{e^{-st}}{-s}) = \int_0^a d(t \cdot \frac{e^{-st}}{-s}) - \frac{e^{-st}}{-s}\, dt.$$

Thus

$$\mathcal{L}(f) = a \cdot \frac{e^{-as}}{-s} + \frac{1}{s}\int_0^a e^{-st}\, dt.$$

Since

$$\int_0^a e^{-st}\, dt = \int_0^a d(\frac{e^{-st}}{-s}) = \frac{e^{-as}}{-s} - \frac{0}{-s} = \frac{1}{s} - \frac{e^{-as}}{s},$$

we have

$$\mathcal{L}(f) = a \cdot \frac{e^{-as}}{-s} + \frac{1}{s^2} - \frac{e^{-as}}{s^2}.$$

Please choose the method you like.

Sometimes it is easy to make a mistake when applying a formula: Please find where we go wrong in the following computations:

$$u(t-\pi)\sin t = u(t-\pi)(-\sin(t-\pi)) \Rightarrow \mathcal{L}(u(t-\pi)\sin t) = -\frac{e^{\pi s}}{s^2+1}.$$

Please compute by yourself if you think it is right !

It is nice to apply formulas when solving differential–integral equations: Consider the following equation for **RLC-Circuit** (see page 29 section 1.5 and page 93 section 2.9 of Kreyszig's book):

$$Li'(t) + Ri(t) + \frac{1}{C}\int_0^t i(\tau)\,d\tau = V(t).$$

Assume that

$$R = C = L = 1,\ V(t) = \delta(t-1),\ i(0) = 0.$$

Then we have

$$\mathcal{L}(i') + \mathcal{L}(i) + \mathcal{L}(\int_0^t i(\tau)) = \mathcal{L}(\delta(t-1)).$$

Apply the formulas, if you are lucky then you can get

$$sI - 0 + I + \frac{I}{s} = e^{-s},$$

i.e.

$$I = \frac{e^{-s}}{s+1+\frac{1}{s}} = e^{-s}\frac{s}{s^2+s+1}.$$

Then we know that

$$i(t) = \mathcal{L}^{-1}\left(e^{-s}\frac{s}{s^2+s+1}\right).$$

Exercise: Compute $\mathcal{L}^{-1}\left(e^{-s}\frac{s}{s^2+s+1}\right)$.

11. Convolution, $\mathcal{L}(f\star g) = \mathcal{L}(f)\cdot\mathcal{L}(g)$

Let $f(t), g(t)$ be two functions for $t \geq 0$.

Definition 11.1 (Convolution of f and g)**.**

$$(f\star g)(t) := \int_0^t f(\tau)g(t-\tau)\,d\tau, \qquad t \geq 0.$$

Example: $1\star t = \frac{t^2}{2}$: we have

$$1\star t = \int_0^t 1\cdot(t-\tau)\,d\tau = t^1 - \frac{t^2}{2} = \frac{t^2}{2}.$$

Example: $t\star 1 = \frac{t^2}{2}$: we have

$$t\star 1 = \int_0^t \tau\,d\tau = \frac{t^2}{2}.$$

In general, consider $\tau = t-u$, we have

$$\int_0^t f(\tau)g(t-\tau)\,d\tau = \int_0^t f(t-u)g(u)\,d(u) = \int_0^t g(u)f(t-u)\,du,$$

which gives

$$f\star g = g\star f.$$

Exercise:

$$e^t\star e^t = te^t,$$

$$f(t) \star 1 = \int_0^t f(\tau)\, d\tau,$$

$$f(t) \star \delta(t) = f(t), \;\; f(t) \star \delta(t-a) = u(t-a) f(t-a).$$

Theorem 11.2 (Laplace transform of convolution)**.**

$$\mathcal{L}(f \star g) = \mathcal{L}(f) \cdot \mathcal{L}(g).$$

Proof. [Not assumed in this course]. We have

$$\mathcal{L}(f \star g) = \int_0^\infty e^{-st} \left(\int_0^t f(\tau) g(t-\tau)\, d\tau \right) dt.$$

Since

$$\{(t,\tau) : 0 < t < \infty, \; 0 < \tau < t\} = \{(t,\tau) : 0 < \tau < \infty, \; t > \tau\},$$

we have

$$\int_0^\infty e^{-st} \left(\int_0^t f(\tau) g(t-\tau)\, d\tau \right) dt = \int_0^\infty f(\tau) \left(\int_\tau^\infty e^{-st} g(t-\tau)\, dt \right) d\tau.$$

Notice that if we take $t - \tau = x$ then

$$\int_\tau^\infty e^{-st} g(t-\tau)\, dt = \int_0^\infty e^{-s(\tau+x)} g(x)\, dx = e^{-s\tau} \mathcal{L}(g).$$

Now we have

$$\int_0^\infty f(\tau) \left(\int_\tau^\infty e^{-st} g(t-\tau)\, dt \right) d\tau = \left(\int_0^\infty f(\tau) e^{-s\tau}\, d\tau \right) \cdot \mathcal{L}(g),$$

which gives

$$\mathcal{L}(f \star g) = \mathcal{L}(f) \cdot \mathcal{L}(g).$$

□

Remark 1: Since $f(t) \star \delta(t-a) = u(t-a) f(t-a)$, the above theorem gives

$$\mathcal{L}(f) e^{-as} = \mathcal{L}(u(t-a) f(t-a)).$$

Remark 2: Since

$$\mathcal{L}(f \star (g \star h)) = \mathcal{L}(f) \cdot \mathcal{L}(g \star h) = \mathcal{L}(f) \cdot \mathcal{L}(g) \cdot \mathcal{L}(h) = \mathcal{L}((f \star g) \star h),$$

we get $f \star (g \star h) = (f \star g) \star h$ (can you prove this directly ?).

Compute: $t^m \star t^n = \frac{m!n!}{(m+n+1)!} t^{m+n+1}, \; m, n = 0, 1, \cdots$ The above theorem gives

$$t^m \star t^n = \mathcal{L}^{-1} \mathcal{L}(t^m \star t^n) = \mathcal{L}^{-1}\left(\frac{m!}{s^{m+1}} \cdot \frac{n!}{s^{n+1}}\right).$$

Example $\mathcal{L}^{-1}(\frac{1}{(s^2+1)^2}) = \frac{\sin t - t \cos t}{2}$. Since

$$\mathcal{L}^{-1}\left(\frac{1}{s^2+1}\right) = \sin t,$$

we have

$$\mathcal{L}^{-1}\left(\frac{1}{(s^2+1)^2}\right) = \mathcal{L}^{-1}(\mathcal{L}(\sin t) \cdot \mathcal{L}(\sin t)) = \sin t \star \sin t = \int_0^t \sin(\tau) \sin(t-\tau)\, d\tau.$$

By (3), we have

$$2\sin(\tau)\sin(t-\tau) = \cos(\tau-(t-\tau)) - \cos(\tau+(t-\tau)) = \cos(2\tau-t) - \cos t.$$

Thus

$$\int_0^t \sin(\tau)\sin(t-\tau)\,d\tau = \int_0^t \frac{\cos(2\tau-t)-\cos t}{2}\,d\tau = \frac{\sin t - \sin(-t)}{4} - \frac{t\cos t}{2},$$

and we have

$$\mathcal{L}^{-1}(\frac{1}{(s^2+1)^2}) = \frac{\sin t - t\cos t}{2}.$$

Example: differential equation: Consider

$$y'' + y = \sin t,\ y(0) = 0,\ y'(0) = 1.$$

Apply the Laplace transform, we get

$$s^2Y - sy(0) - y'(0) + Y = \mathcal{L}(\sin t),$$

i.e.

$$s^2Y - 1 + Y = \frac{1}{s^2+1}.$$

We have

$$Y = \frac{1}{s^2+1} + \frac{1}{(s^2+1)^2}.$$

By the above example, we have

$$y = \sin t + \frac{\sin t - t\cos t}{2} = \frac{3\sin t - t\cos t}{2}.$$

Example: Convolution equation: Consider

$$y - \int_0^t (t-\tau)y(\tau)\,d\tau = 1.$$

Notiec that

$$\int_0^t (t-\tau)y(\tau)\,d\tau = y \star t.$$

Apply Laplace transform to the equation, we get

$$Y - Y\cdot\frac{1}{s^2} = \frac{1}{s},$$

i.e.

$$Y = \frac{1}{s(1-s^{-2})} = \frac{s}{s^2-1} = \frac{1}{2}\left(\frac{1}{s-1} + \frac{1}{s+1}\right).$$

Thus

$$y = \frac{e^t + e^{-t}}{2} = \cosh t.$$

Application to non-homogeneous linear ODEs: Consider

$$y'' + by' + cy = r(t),$$

given $y(0)$ and $y'(0)$, we have

$$s^2Y - sy(0) - y'(0) + b(sY - y(0)) + cY = R(s).$$

Thus

$$Y = \frac{1}{s^2+bs+c} \cdot R(s) + \frac{sy(0)+y'(0)+by(0)}{s^2+bs+c} := K(s)\cdot R(s) + G(s).$$

Thus

$$y = k \star r + g.$$

Example: Consider

$$y'' + y = r(t), \;\; y(0) = y'(0) = 0.$$

Apply the Laplcae transform, we have

$$s^2 Y + Y = \mathcal{L}(r).$$

Thus

$$Y = \frac{1}{s^2+1} \cdot \mathcal{L}(r),$$

which gives

$$y(t) = \sin t \star r.$$

12. $F'(s) = \mathcal{L}(-tf(t))$

Theorem 12.1. *Let $F(s) = \mathcal{L}(f)$. Then we have*

$$F'(s) = \mathcal{L}(-tf(t)),$$

and

$$\int_s^\infty F(u)\,du = \mathcal{L}(\frac{f(t)}{t}), \textit{ not assumed in this course}.$$

Proof. Apply differential to

$$F(s) = \int_0^\infty e^{-st} f(t)\,dt,$$

we get

$$F'(s) = \int_0^\infty \frac{d(e^{-st})}{ds} f(t)\,dt = \int_0^\infty e^{-st}\cdot(-tf(t))\,dt = \mathcal{L}(-tf(t)).$$

For the second formula,

$$\int_s^\infty F(u)\,du = \int_s^\infty \left(\int_0^\infty e^{-ut} f(t)\,dt\right)\,du,$$

change the order of integration, we get

$$\int_s^\infty \left(\int_0^\infty e^{-ut} f(t)\,dt\right)\,du = \int_0^\infty f(t)\left(\int_s^\infty e^{-ut}\,du\right)\,dt.$$

Thus the second formula follows from

$$\int_s^\infty e^{-ut}\,du = \int_s^\infty d(\frac{e^{-ut}}{-t}) = \frac{e^{-st}}{t}.$$

□

Compute $\mathcal{L}(t\sin t) = \frac{2s}{(s^2+1)^2}$**:** By the above theorem,

$$\mathcal{L}(t\sin t) = -(\frac{1}{s^2+1})' = \frac{2s}{(s^2+1)^2}.$$

Compute: $\mathcal{L}^{-1}(\ln(1+s^{-2})) = \frac{2-2\cos t}{t}$: Let $\ln(1+s^{-2}) = F(s) = \mathcal{L}(f)$, then

$$F' = (\ln(1+s^2) - \ln(s^2))' = \frac{2s}{1+s^2} - \frac{2}{s}.$$

Thus

$$\mathcal{L}^{-1}(F') = 2\cos t - 2.$$

By the above theorem, we have

$$\mathcal{L}^{-1}(F') = -tf(t).$$

thus

$$f(t) = \frac{2-2\cos t}{t}.$$

13. System of differential equations

We shall only give an example:

$$y_1' = -y_1 + y_2; \ \ y_2' = -y_1 - y_2 + f(t), \ \ y_1(0) = y_2(0) = 0.$$

Apply the Laplace transform, we get

$$sY_1 = -Y_1 + Y_2; \ \ sY_2 = -Y_1 - Y_2 + F(s).$$

Thus

$$(s+1)Y_1 - Y_2 = 0;$$

and

$$Y_1 + (s+1)Y_2 = F(s).$$

The first one gives $Y_2 = (s+1)Y_1$, together with the second, we have

$$Y_1 = F(s)(1+(s+1)^2)^{-1}.$$

Thus

$$Y_2 = F(s)(s+1)(1+(s+1)^2)^{-1}.$$

Now we have

$$y_1 = f(t) \star (e^{-t}\sin t), \ y_2 = f(t) \star (e^{-t}\cos t).$$

when $f(t) = e^{-t}$, we get

$$y_1 = \int_0^t e^{-(t-\tau)} e^{-\tau} \sin\tau \, d\tau = e^{-t}(1-\cos t), \ \ y_2 = e^{-t}\sin t.$$

14. COMPLEX FOURIER SERIES

Fix $p > 0$, if

$$f(x+p) = f(x), \ \forall \, x \in \mathbb{R},$$

then we call f a *periodic function with period* p.

Example: periodic function:

1. A polynomial is periodic if and only if it is a constant;
2. $e^{\lambda x}$ is has period 2π if and only if

$$\lambda = in, \ \ n \in \mathbb{Z}.$$

Recall that the main idea of this course is to represent a function f by eigenvectors $e^{\lambda x}$ of the derivative. If f has period 2π then we hope that those eigenvectors that have the same period 2π will be enough to represent f. The main theorem in Fourier analysis is the following:

Theorem 14.1 (Fourier 1807)**.** *If f has period 2π and is smooth enough then we have*

$$f(x) = \sum_{n \in \mathbb{Z}} c_n \, e^{inx}, \ \forall \, x \in \mathbb{R}.$$

——*The proof (see Page 63 in* [3]*) is not assumed in this course.*

What does "smooth enough" mean? It means that f is piecewise smooth and

$$f(x_0) = \frac{f(x_0+) + f(x_0-)}{2},$$

if f is not smooth at x_0.

How to compute c_n ? We shall prove that

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} \, dx.$$

In fact , by the above theorem, we have

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} \, dx = \sum_{m \in \mathbb{Z}} \frac{c_m}{2\pi} \int_{-\pi}^{\pi} e^{imx} e^{-inx} \, dx.$$

By (2), we have

$$e^{imx} e^{-inx} = e^{i(m-n)x}.$$

If $m = n$ then it gives

$$\int_{-\pi}^{\pi} e^{imx} e^{-inx} \, dx = \int_{-\pi}^{\pi} 1 \, dx = 2\pi.$$

Notice that if $m \neq n$ then we have

$$\int_{-\pi}^{\pi} e^{i(m-n)x} \, dx = \int_{-\pi}^{\pi} d\left(\frac{e^{i(m-n)x}}{i(m-n)}\right) = 0.$$

Thus

$$\sum_{m \in \mathbb{Z}} \frac{c_m}{2\pi} \int_{-\pi}^{\pi} e^{imx} e^{-inx} \, dx = c_n.$$

Definition 14.2. *We call*

$$f(x) = \sum_{n \in \mathbb{Z}} c_n e^{inx}$$

the ***complex Fourier series*** *of f and*

$$c_n := \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx}\, dx,\ n \in \mathbb{Z},$$

the ***complex Fourier coefficients*** *of f.*

Remark: (not assumed in this course): The above theorem is also true for periodic delta function

$$f = \sum_{k \in \mathbb{Z}} \delta(x - 2k\pi),$$

then we have

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} \delta(x) e^{-inx}\, dx = \frac{1}{2\pi},$$

thus

$$\sum_{k \in \mathbb{Z}} \delta(x + 2k\pi) = \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} e^{inx}.$$

This formula says that the spectruml set of the periodic delta function is $\mathbb{Z}$, which explains the *Poisson summation formula*.

Example: Consider

$$f(x) = 1,\ 0 < x < \pi;\ \ f(x) = -1,\ -\pi < x < 0,$$

and

$$f(0) = f(\pi) = f(-\pi) = 0.$$

Then we know f is smooth enough and

$$2\pi c_n = \int_{-\pi}^{\pi} f(x) e^{-inx}\, dx = \int_0^{\pi} e^{-inx}\, dx - \int_{-\pi}^{0} e^{-inx}\, dx.$$

Since

$$\int_0^{\pi} e^{-inx}\, dx = \int_0^{\pi} d(\frac{e^{-inx}}{-in}) = \frac{(-1)^n - 1}{-in},$$

and

$$\int_{-\pi}^{0} e^{-inx}\, dx = \int_{-\pi}^{0} d(\frac{e^{-inx}}{-in}) = \frac{1 - (-1)^n}{-in},$$

we have

$$2\pi c_n = \frac{2(1 - (-1)^n)}{in},$$

i.e.

$$c_n = \frac{2}{in\pi},\ n\ odde;\ \ c_n = 0,\ n\ even.$$

Thus the complex fourier series of f is

$$f(x) = \sum_{m \in \mathbb{Z}} \frac{2}{i(2m+1)\pi} e^{i(2m+1)x}.$$

15. (REAL) FOURIER SERIES

In the previous example, f is a real function, thus the complex Fourier series should also be a real function. Let us verify this fact:

$$f(x) = \frac{2}{i\pi}\left(e^{ix} + \frac{e^{3ix}}{3} + \cdots\right) + \frac{2}{i\pi}\left(\frac{e^{-ix}}{-1} + \frac{e^{-3ix}}{-3} + \cdots\right),$$

thus

$$f(x) = \frac{2}{i\pi}\left((e^{ix} - e^{-ix}) + \frac{e^{3ix} - e^{-3ix}}{3} + \cdots\right).$$

Since

$$e^{inx} - e^{-inx} = 2i \sin nx,$$

we get

$$f(x) = \frac{4}{\pi}\left(\sin x + \frac{\sin 3x}{3} + \cdots\right),$$

In particular, it gives

$$1 = f(\frac{\pi}{2}) = \frac{4}{\pi}\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\right).$$

Thus

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4},$$

which is a famous formula obtained by Leibniz in 1673 from geometric considerations.

Fourier series: In general, by the Euler formula

$$e^{inx} = \cos nx + i \sin nx,$$

we know that

$$\sum c_n e^{e^{inx}} = \sum c_n(\cos nx + i \sin nx),$$

which gives

$$f(x) = c_0 + \sum_{n=1}^{\infty} c_n(\cos nx + i \sin nx) + \sum_{n=1}^{\infty} c_{-n}(\cos nx - i \sin nx).$$

Thus we have

$$f(x) = c_0 + \sum_{n=1}^{\infty} \left((c_n + c_{-n}) \cos nx + i(c_n - c_{-n}) \sin nx\right),$$

Recall that

$$(c_n + c_{-n}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)(e^{-inx} + e^{inx})\, dx = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx\, dx,$$

and

$$i(c_n - c_{-n}) = \frac{i}{2\pi}\int_{-\pi}^{\pi} f(x)(e^{-inx} - e^{inx})\, dx = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx\, dx,$$

thus we get the following theorem:

Theorem 15.1. *If f has period 2π and is smooth enough then it has the following **Fourier series** expansion*

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx),$$

*where a_0, a_n, b_n are the **Fourier coefficients** of f such that*

$$a_0 = c_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx,$$

and for $n = 1, 2 \cdots$, we have

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\, dx,$$

and

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\, dx.$$

Remark: We can aslo compute a_n, b_n directly by using the following proposition.

Proposition 15.2. *Put $\delta_{mn} = 1$ if $m = n$ and $\delta_{mn} = 0$ if $m \neq n$ then*

$$\int_{-\pi}^{\pi} \cos nx \cos mx\, dx = \int_{-\pi}^{\pi} \sin nx \sin mx\, dx = \pi \delta_{mn}, \quad m, n = 1, 2, \cdots,$$

and

$$\int_{-\pi}^{\pi} \cos nx\, dx = 2\pi \delta_{n0}, \quad \int_{-\pi}^{\pi} \cos nx \sin mx = 0, \quad m, n = 0, 1, 2, \cdots.$$

Proof. Follows from the Euler formula

$$\cos nx = \frac{e^{inx} + e^{-inx}}{2}, \ \sin nx = \frac{e^{inx} - e^{-inx}}{2i},$$

and

$$\int_{-\pi}^{\pi} e^{inx} e^{-imx}\, dx = 2\pi \delta_{mn}.$$

□

Exercise: Using the above proposition to prove the formulas for a_n, b_n.

Example: Consider

$$f(x) = 0, \quad -\pi < x < 0; \quad f(x) = x, \quad 0 \leq x < \pi.$$

Then

$$2\pi a_0 = \int_{-\pi}^{\pi} f(x)\, dx = \int_0^{\pi} x\, dx = \frac{\pi^2}{2},$$

and

$$\pi a_n = \int_{-\pi}^{\pi} f(x) \cos nx\, dx = \int_0^{\pi} x \cos nx\, dx = \int_0^{\pi} x\, d(\frac{\sin nx}{n}) = -\int_0^{\pi} \frac{\sin nx}{n}\, dx.$$

Since

$$-\int_0^{\pi} \frac{\sin nx}{n}\, dx = \int_0^{\pi} d(\frac{\cos nx}{n^2}) = \frac{(-1)^n - 1}{n^2},$$

we get

$$a_0 = \frac{\pi}{4}, \quad a_{2m} = 0, \ a_{2m-1} = \frac{-2}{(2m-1)^2 \pi}, \quad m = 1, 2 \cdots.$$

Moreover, we have

$$\pi b_n = \int_{-\pi}^{\pi} f(x)\sin nx\,dx = \int_0^{\pi} x\sin nx\,dx = \int_0^{\pi} x\,d(\frac{-\cos nx}{n}) = \frac{\pi(-1)^{n+1}}{n} + \int_0^{\pi}\frac{\cos nx}{n}\,dx,$$

Notice that

$$\int_0^{\pi}\frac{\cos nx}{n}\,dx = \int_0^{\pi} d(\frac{\sin nx}{n^2}) = 0,$$

thus

$$b_n = \frac{(-1)^{n+1}}{n}.$$

Thus

$$f(x) = \frac{\pi}{4} - \frac{2}{\pi}\left(\cos x + \frac{\cos 3x}{3^2} + \cdots\right) + \left(\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \cdots\right).$$

Take $x = 0$ then we get

$$0 = \frac{\pi}{4} - \frac{2}{\pi}(1 + \frac{1}{3^2} + \cdots),$$

i.e.

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}.$$

Exercise: Use $1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$ to prove that

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}.$$

16. Odd or Even extension: Fourier Sine and Cosine series

Definition 16.1. *We say that f is odd if $f(-x) = -f(x)$; f is even if $f(-x) = f(x)$.*

Example: For every positive integer n, we know that $\cos nx$ is even and $\sin nx$ is odd.

Application: If f is even then

$$\int_{\pi}^{\pi} f(x)\,dx = 2\int_0^{\pi} f(x)\,dx.$$

If f is odd then

$$\int_{\pi}^{\pi} f(x)\,dx = 0.$$

In particular, if f is odd then all

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = 0;$$

if f is even then all

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = 0.$$

Thus we proved the following theorem:

Theorem 16.2. *Assume that f has period 2π and is smooth enough. If f is **odd** then it can be written as a **Fourier sine series***

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx, \;\; b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx \, dx.$$

*If f is **even** then it can be written as a **Fourier cosine series***

$$f(x) = \frac{1}{\pi}\int_0^{\pi} f(x)\, dx + \sum_{n=1}^{\infty} a_n \cos nx, \; a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx \, dx.$$

Odd or Even extension: Let f be a function in $(0, \pi)$. Then we can extend f to an odd function, say f_o such that

$$f_o(-x) = -f(x), \; x \in (0, \pi);$$

we can also extend f to an even function, say f_e such that

$$f_e(-x) = f(x), \; x \in (0, \pi).$$

Example: Consider a function

$$f(x) = x, \;\; 0 < x < \frac{\pi}{2}; \;\; f(x) = \frac{\pi}{2}, \;\; \frac{\pi}{2} < x < \pi,$$

then we can write

$$f_o(x) = \sum b_n \sin nx,$$

and

$$f_e(x) = a_0 + \sum a_n \cos nx.$$

By the above theorem, we have

$$\pi a_0 = \int_0^{\pi} f(x)\, dx = \int_0^{\frac{\pi}{2}} x\, dx + \int_{\frac{\pi}{2}}^{\pi} \frac{\pi}{2}\, dx = \frac{1}{2}(\frac{\pi}{2})^2 + \frac{\pi}{2}(\pi - \frac{\pi}{2}) = \frac{3\pi^2}{8},$$

and

$$\frac{\pi}{2} a_m = \int_0^{\pi} f(x)\cos mx\, dx = \int_0^{\frac{\pi}{2}} x\cos mx\, dx + \int_{\frac{\pi}{2}}^{\pi} \frac{\pi}{2}\cos mx\, dx.$$

Since $\int_0^{\frac{\pi}{2}} x\cos mx\, dx$ can be written as

$$\int_0^{\frac{\pi}{2}} x\, d(\frac{\sin mx}{m}) = \frac{\pi}{2m}\sin\frac{m\pi}{2} - \int_0^{\frac{\pi}{2}} \frac{\sin mx}{m} dx = \frac{\pi}{2m}\sin\frac{m\pi}{2} + \frac{1}{m^2}(\cos\frac{m\pi}{2} - 1),$$

and

$$\int_{\frac{\pi}{2}}^{\pi} \frac{\pi}{2}\cos mx\, dx = -\frac{\pi}{2m}\sin\frac{m\pi}{2},$$

we get

$$a_m = \frac{2}{m^2\pi}(\cos\frac{m\pi}{2} - 1).$$

Thus

$$f_e(x) = \frac{3\pi}{8} + \frac{2}{\pi}\left(-\cos x - \frac{2\cos 2x}{2^2} - \frac{\cos 3x}{3^2} - \frac{\cos 5x}{5^2} - \cdots\right).$$

For b_n we have

$$\frac{\pi}{2}b_m = \int_0^\pi f(x)\sin mx\,dx = \int_0^{\frac{\pi}{2}} x\sin mx\,dx + \int_{\frac{\pi}{2}}^\pi \frac{\pi}{2}\sin mx\,dx.$$

Since $\int_0^{\frac{\pi}{2}} x\sin mx\,dx$ can be written as

$$\int_0^{\frac{\pi}{2}} x\,d(\frac{\cos mx}{-m}) = \frac{-\pi}{2m}\cos\frac{m\pi}{2} + \int_0^{\frac{\pi}{2}} \frac{\cos mx}{m}dx = \frac{-\pi}{2m}\cos\frac{m\pi}{2} + \frac{1}{m^2}\sin\frac{m\pi}{2},$$

and

$$\int_{\frac{\pi}{2}}^\pi \frac{\pi}{2}\sin mx\,dx = -\frac{\pi}{2m}(\cos m\pi - \cos\frac{m\pi}{2}),$$

we have

$$b_m = \frac{2}{\pi}\left(\frac{1}{m^2}\sin\frac{m\pi}{2} - \frac{\pi}{2m}\cos m\pi\right) = \frac{2\sin\frac{m\pi}{2}}{m^2\pi} - \frac{\cos m\pi}{m}$$

Thus

$$\begin{aligned} f_o(x) &= \left(\frac{2}{\pi}+1\right)\sin x + \left(0-\frac{1}{2}\right)\sin 2x + \left(\frac{-2}{3^2\pi}+\frac{1}{3}\right)\sin 3x \\ &\quad + \left(0+\frac{1}{4}\right)\sin 4x + \left(\frac{2}{5^2\pi}+\frac{1}{5}\right)\sin 5x + \cdots . \end{aligned}$$

17. Approximation by Trigonometric polynomials

Let f be a smooth enough function with period 2π. We hope to find an N-series,

$$F_N := \sum_{|n|\leq N} C_n e^{inx},$$

such that

$$\int_{-\pi}^\pi |F_N - f|^2\,dx$$

is minimal. The main idea is to use *orthogonal decomposition*.

Orthogonal Decomposition in vector space: Let S be a subspace of a vector space V, then we can write a vector, say v, in V as

$$v = v_S + v_{S^\perp},$$

where v_S lies in S and $v_{S^\perp}$ is orthoginal to S. Then it is very clear from the picture that v_S is the unique solution of the following extremal problem:

$$||v_S - v|| = \min\{||u-v|| : u \in S\}.$$

For a real proof it is enough to use

$$||u-v||^2 = ||u-v_S||^2 + ||v_{S^\perp}||^2,$$

which implies that $u = v_S$ is the unique solution.

Orthogonal Decomposition in L^2-space: In our case, we consider V as the space of *complex* functions spanned by $\{e^{inx}\}_{n\in\mathbb{Z}}$ with the following inner product structure:

$$(f,g) := \int_{-\pi}^\pi f(x)\overline{g(x)}\,dx, \;\; ||f||^2 := (f,f).$$

Now S is the subspace of V spanned by e^{inx}, $|n| \leq N$. Let f be a smooth enough function with period 2π. Then we know that

$$||f_S - f|| = \min\{||u - f|| : u \in S\}$$

Thus $F_N = f_S$ solves our extremal problem.

What is f_S ? The simplest way is to use the complex Fourier series expansion

$$f(x) = \sum_{n \in \mathbb{Z}} c_n e^{inx},$$

Put

$$f_N = \sum_{|n| \leq N} c_n e^{inx},$$

since $\{e^{inx}\}_{n \in \mathbb{Z}}$ is an orthogonal basis of V we have

$$(f_N, f - f_N) = 0,$$

which implies that

$$f_N = f_S.$$

Thus we have

Theorem 17.1. *Complex Fourier series expansion solves the best trigonometric polynomial approximation problem.*

Bessel's inequality and Parseval's identity: Notice that

$$||f||^2 = ||f_N||^2 + ||f - f_N||^2,$$

Thus we get the Bessel inequality

$$||f||^2 \geq ||f_N||^2,$$

i.e.

$$\int_{-\pi}^{\pi} |f(x)|^2 \, dx \geq 2\pi \cdot \sum_{|n| \leq N} |c_n|^2,$$

and the Parseval identity

$$\int_{-\pi}^{\pi} |f(x)|^2 \, dx = 2\pi \cdot \sum_{n \in \mathbb{Z}} |c_n|^2.$$

Example: Consider the example in the end of section 14:

$$f(x) = 1,\ 0 < x < \pi;\ \ f(x) = -1,\ -\pi < x < 0,$$

and

$$f(0) = f(\pi) = f(-\pi) = 0.$$

We know that f has the following complex Fourier series expansion:

$$f(x) = \sum_{m \in \mathbb{Z}} \frac{2}{i(2m+1)\pi} e^{i(2m+1)x}.$$

Thus the Parseval identity gives

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 \, dx = 1 = \sum_{m \in \mathbb{Z}} \frac{4}{(2m+1)^2 \pi^2} = \frac{8}{\pi^2} \left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots \right),$$

which gives another proof of

$$1+\frac{1}{3^2}+\frac{1}{5^2}+\cdots=\frac{\pi^2}{8}.$$

18. Fourier transform: basic facts

Definition 18.1. *We call*

$$\hat{f}(w):=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-iwx}\,dx,$$

the ***Fourier transform*** *of f and write $\hat{f}=\mathcal{F}(f)$.*

Example: F1: Fourier transform of $f(x)=1$ if $|x|<1$ and $f(x)=0$ otherwise:

$$\hat{f}(w):=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-iwx}\,dx=\frac{1}{\sqrt{2\pi}}\int_{-1}^{1} e^{-iwx}\,dx$$

if $w\neq 0$ then

$$\int_{-1}^{1} e^{-iwx}\,dx=\int_{-1}^{1} d(\frac{e^{-iwx}}{-iw})=\frac{e^{-iw}}{-iw}-\frac{e^{iw}}{-iw}=\frac{2\sin w}{w}.$$

Notice that

$$\lim_{w\to 0}\frac{2\sin w}{w}=2=\int_{-1}^{1} dx=\hat{f}(0).$$

Thus we can write

$$\hat{f}(w)=\frac{1}{\sqrt{2\pi}}\left(\frac{2\sin w}{w}\right)=\sqrt{\frac{2}{\pi}}\frac{\sin w}{w}.$$

Example: F2: Fourier transform of $f(x)=e^{-x}$ if $x>0$ and $f(x)=0$ otherwise:

$$\hat{f}(w):=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-iwx}\,dx=\frac{1}{\sqrt{2\pi}}\int_{0}^{\infty} e^{-x}e^{-iwx}\,dx=\frac{1}{\sqrt{2\pi}}\mathcal{L}(e^{-iwt})(1).$$

Recall that

$$L(e^{-iwt})(s)=\frac{1}{s+iw},$$

thus

$$\hat{f}(w)=\frac{1}{\sqrt{2\pi}}\cdot\frac{1}{1+iw}.$$

From Complex Fourier series to inverse Fourier transform: Assume that f is smooth enough in $-N<x<N$ and $f=0$ when $|x|>N$. For each $L>N$, let us define a periodic function f_L such that

$$f_L(x)=f(x),\ \ |x|<L;\ \ f_L(x+2L)=f_L(x).$$

Then we know that

$$g_L(x)=f_L\left(\frac{Lx}{\pi}\right),$$

has period p and is smooth enough. Thus

$$g_L(x)=\sum c_n e^{inx},\ \ c_n=\frac{1}{2\pi}\int_{-\pi}^{\pi} g_L(x)e^{-inx}\,dx.$$

Thus

$$f_L(x)=g_L\left(\frac{\pi x}{L}\right)=\sum c_n e^{in\frac{\pi x}{L}}.$$

Consider $v = \frac{Lx}{\pi}$, we can write

$$c_n = \frac{1}{2\pi}\int_{-L}^{L} f_L(v)e^{-in\frac{\pi v}{L}}\, d(\frac{\pi v}{L}) = \frac{1}{2L}\int_{-\infty}^{\infty} f(v)e^{-in\frac{\pi v}{L}}\, dv = \frac{\sqrt{2\pi}}{2L}\hat{f}(\frac{n\pi}{L}).$$

which gives

$$f(x) = \sqrt{\frac{\pi}{2}}\sum_{n\in\mathbb{Z}} \frac{\hat{f}(\frac{n\pi}{L})\cdot e^{in\frac{\pi x}{L}}}{L}.$$

Put

$$\Delta w = \frac{\pi}{L},$$

then we have

$$f(x) = \frac{1}{\sqrt{2\pi}}\sum_{n\in\mathbb{Z}} \hat{f}(n\Delta w)\cdot e^{ix\cdot n\Delta w}\Delta w.$$

Assume that $\hat{f}(w)e^{ixw}$ is integrable in $-\infty < x < \infty$. Let L goes to infty, the above formula gives

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(w)e^{ixw}\, dw.$$

Definition 18.2 (Fourier inversion formula)**.** *If*

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(w)e^{ixw}\, dw, \tag{10}$$

then we say that $f(x)$ is the ***inverse Fourier transform*** *of $\hat{f}(w)$ and write $f = \mathcal{F}^{-1}(\hat{f})$.*

When is the Fourier inversion formula (10) **true ?**

1. It is known that (see Page 141 Theorem 1.9 in [3]) the Fourier inversion formula is true is true if f is *smooth and rapidly decreaing*, in the sense that

$$\sup_{x\in\mathbb{R}} |x|^k|f^{(l)}(x)| < \infty, \quad \text{for every } k, l \geq 0,$$

where $f^{(l)}$ denotes the l-th derivative of f.

2. (Not assumed in this course) In general, assume that f is *good*: i.e. f is piecewise smooth, $\int_{\mathbb{R}} |f|\, dx < \infty$ and

$$f(x_0) = \frac{f(x_0+) + f(x_0-)}{2}.$$

if f is not smooth at x_0. Then Fourier inversion formula is true in the following sense (see Page 171, Theorem 7.1 in [4])

$$f(x) = \lim_{L\to\infty} \frac{1}{\sqrt{2\pi}}\int_{-L}^{L} \hat{f}(w)e^{ixw}\, dw.$$

Example: $f(x) = e^{-\frac{x^2}{2}}$ **is smooth and rapidly decreasing**. Let us compute

$$\hat{f}(w) := \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}e^{-iwx}dx.$$

The idea is look at the derivative of $\hat{f}(w)$:

$$\hat{f}'(w) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}(-ix)e^{-iwx}dx = \mathcal{F}(-ixf(x)). \tag{11}$$

Notice that $(e^{-\frac{x^2}{2}})' = e^{-\frac{x^2}{2}}(-x)$, thus

$$\hat{f}'(w) = \frac{i}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-iwx} d(e^{-\frac{x^2}{2}}) = \frac{-i}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} d(e^{-iwx}) = -w\hat{f}(w).$$

Now we have

$$\left(\hat{f}(w)e^{\frac{w^2}{2}}\right)' = (-w+w)\left(\hat{f}(w)e^{\frac{w^2}{2}}\right) = 0.$$

Thus $\hat{f}(w)e^{\frac{w^2}{2}}$ is a constant, i.e.

$$\hat{f}(w)e^{\frac{w^2}{2}} \equiv \hat{f}(0)e^0 = \hat{f}(0).$$

Now we have

$$\hat{f}(w) = \hat{f}(0)e^{\frac{-w^2}{2}} = \hat{f}(0)f(w)$$

The above theorem implies that

$$f(x) = \mathcal{F}^{-1}(\hat{f}) = \hat{f}(0)\mathcal{F}^{-1}(f) = \hat{f}(0)\hat{f}(-x) = (\hat{f}(0))^2 f(x).$$

Since

$$\hat{f}(0) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}\, dx > 0,$$

we get

$$\hat{f}(0) = 1, \quad \hat{f} = f.$$

Remark: normal distribution: One may also use integration on $\mathbb{R}^2$ to compute the following integral directly (see page 138 formula (6) in [3])

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}\, dx = 1. \tag{12}$$

Consider

$$u = \sqrt{t}x + \mu, \quad t > 0, \quad \mu \in \mathbb{R},$$

the above formula implies the following classical formula in Gauss's normal distribution theory

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi t}}\, e^{-\frac{(u-\mu)^2}{2t}}\, du = 1, \tag{13}$$

where

$$f(x\,|\,\mu, t) := \frac{1}{\sqrt{2\pi t}}\, e^{-\frac{(u-\mu)^2}{2t}}, \tag{14}$$

is the *probability density of the normal distribution with expectation* μ *and variance* t.

Examples of Fourier inversion formula for "good" functions, not assumed in this course:

1. Consider the function in **Example: F1**, the Fourier inversion formula gives

$$f(x) = \lim_{L\to\infty}\frac{1}{\sqrt{2\pi}}\int_{-L}^{L}\sqrt{\frac{2}{\pi}}\frac{\sin w}{w}e^{ixw}\, dw = \lim_{L\to\infty}\frac{2}{\pi}\int_{0}^{L}\frac{\sin w}{w}\cos wx\, dx.$$

where $f(x) = 1$ if $|x| < 1$, $f(x) = 0$ if $|x| > 1$ and $f(x) = \frac{1}{2}$ if $|x| = 1$. In particular, take $x = 0$ we get

$$\frac{\pi}{2} = \lim_{L\to\infty}\int_{0}^{L}\frac{\sin w}{w}\, dx$$

2. Consider the function in **Example: F2**, the Fourier inversion formula gives

$$f(x) = \lim_{L\to\infty} \frac{1}{2\pi} \int_{-L}^{L} \frac{1}{1+iw} e^{ixw}\, dw,$$

where $f(x) = e^{-x}$ if $x > 0$, $f(x) = 0$ if $x < 0$ annd $f(0) = \frac{1}{2}$. Take $x = 0$, we have

$$\frac{1}{2} = \lim_{L\to\infty} \frac{1}{2\pi} \int_{-L}^{L} \frac{1}{1+iw}\, dw = \frac{1}{\pi} \int_{0}^{\infty} \frac{1}{1+w^2}\, dw,$$

i.e.

$$\int_0^\infty \frac{1}{1+w^2}\, dw = \frac{\pi}{2}.$$

Exercise: Use $w = \tan\theta := \frac{\sin\theta}{\cos\theta}$ to prove the last integreal.

19. Fourier transform of derivative and convolution, discrete Fourier transform ?

In this section, we only consider functions that are smooth and rapidly decreasing.

Following Berndtsson's notes, the text book and the video (21) I will add discrete FT. For the FFT, will ask Anne about discrete Fourier transform and fast Fourier transform.

20. Partial derivatives and Gauss's divergence theorem

use the divergence theorem of Gauss to derive the wave equation and the the heat equation. A3.2, 10.7, 12.1, 12.2, 12.4.

Show that the normal distribution function satisfies the Heat equation. Define the heat kernel.

21. Wave equation

Separating variables, use of Fourier series 12.3

22. Heat equation, I

1. Separating variables, use of Fourier series 12.6
2. Steady case 12.6 , Laplace equation, Separating variables, use of Fourier series

23. Heat equation, II

3. Final 12.7. Modeling Very long bars, solution by Fourier integrals and transforms.

24. Solution of PDEs by Laplace transforms or repetition

Exam (current version):

1 (big problem). Laplace transform, solve second order ODE, use inverse transform.

2. (small). Compute complex Fourier series Given f, if f is sum of cn e inx, compute cn, if cos sin compute an bn.

3. (big) Fourier transform of the gauss function, prove normal distribution integral one, prove that it satisfies the heat equation.

REFERENCES

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[2] E. Kreyszig, *Advanced Engineering Mathematics*, 10 th edition, internation student version.
[3] E. Stein and R. Shakarchi, *Fourier analysis*, Princeton lectures in analysis.
[4] A. Vretblad, *Fourier Analysis and Its Applications*, GTM 233.