Norwegian University of Science and Technology
Department of Mathematical Sciences

TMA4135
Matematikk 4D
Fall 2018

Exercise set 5

$\boxed{1}$ Find a general solution of

$$y'' + \omega^2 y = \cos \beta t, \tag{1}$$

where $\beta \neq \omega$ are positive constants.

Hint: First find the steady state solution (i.e. determine the constants A, B below)

$$y_P(t) := A \sin \beta t + B \cos \beta t,$$

of (1). Then check (or use Laplace transform to show) that

$$y_H = C_1 \cos \omega t + C_2 \sin \omega t,$$

is a general solution of the associated homogeneous equation $y'' + \omega^2 y = 0$. The general solution of (1) is $y = y_P + y_H$.

$\boxed{2}$ Let

$$x^2 = \sum_{n \in \mathbb{Z}} c_n e^{inx},$$

be the complex Fourier series expansion of x^2 on $(-\pi, \pi)$.

a) Compute c_n;

b) Use the Parseval identity for x^2 to calculate the series $\sum_{n=1}^{\infty} \frac{1}{n^4}$.

$\boxed{3}$ Let

$$x^2 = a_0 + \sum_{k=1}^{\infty} \Big(a_k \cos(\frac{kx\pi}{L}) + b_k \sin(\frac{kx\pi}{L}) \Big),$$

be the real Fourier series expansion of x^2 on $(-L, L)$.

a) Compute a_0, a_k, b_k;

b) Let $L = \pi$. Use the result in **a)** to compute the value of the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^3}.$$

Hint: Check the following fact: if

$$f(x) = a_0 + \sum_{k>0} \Big(a_k \cos kx + b_k \sin kx \Big),$$

on $(-\pi, \pi)$ then by term-wise integration

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(x)\, dx = \pi a_0 + \sum_{n=1}^{\infty} 2(-1)^{n+1} \frac{a_{2n-1}}{2n-1}.$$

4 Compute Fourier transforms of the following functions

a)

$$f(x) = \begin{cases} 0 & x \leq 0 \\ e^{-x} & x > 0; \end{cases}$$

b)

$$f(x) = \begin{cases} 1 - x^2 & -1 \leq x \leq 1 \\ 0 & |x| > 1; \end{cases}$$

c)

$$f(x) = \begin{cases} T + x & -T \leq x < 0 \\ T - x & 0 \leq x \leq T \\ 0 & |x| > T. \end{cases}$$

5 Recall the definition of convolution product in Fourier transform theory

$$(f * g)(t) = \int_{-\infty}^{\infty} f(u) g(t-u)\, du.$$

We know that if $f(t) = g(t) = 0$ on $(-\infty, 0)$ then

$$(f * g)(t) = \int_0^t f(\tau) g(t-\tau)\, d\tau,$$

is compatible with the convolution product in Laplace transform theory.

a) For real numbers $a \neq b$, compute the following integral

$$I(a,b) := e^{-at} * e^{-bt} = \int_0^t e^{-a\tau} e^{-b(t-\tau)}\, d\tau;$$

b) Compute the following integral

$$I(a,a) := e^{-at} * e^{-at} = \int_0^t e^{-a\tau} e^{-a(t-\tau)}\, d\tau.$$

Remark: If your computation is correct then you can see that $\lim_{b\to a} I(a,b) = I(a,a)$ gives

$$\lim_{b\to a} \frac{e^{-bt} - e^{-at}}{b-a} = -te^{-at},$$

which is equivalent to the following derivative formula of the exponential function

$$\frac{de^{-at}}{da} = (-t)e^{-at},$$

where we look at e^{-at} as a function of a for every fixed t.