Norwegian University of Science and Technology
Department of Mathematical Sciences

TMA4135
Matematikk 4D
Fall 2018

Exercise set 7

The theory and the codes are taken from the Jupyter notebook on *Polynomial Interpolations.*

Exercises supposed done by hand are marked with (H), exercises in which you are supposed to use/modify code in the Jupyter notebook are marked with an (J). For Jupyter-exercises, hand in a screen-dump of the relevant cell with output.

1 Consider the data points

x_i	-1	$-1/2$	$1/2$	1
$f(x_i)$	$-1/2$	$-5/4$	$1/4$	$5/2$

a) (H) Use Lagrange interpolatation to find the polynomial of minimal degree interpolating these points. Express the polynomial in the form

$$p_n(x) = a_n x^n + \cdots + a_1 x + a_0.$$

Use this to find an approximation to $f(0)$.

b) (J) Plot the polynomial you found on the interval $[-1.5, 1.4]$.
Include a plot of the interpolation points, and confirm by this that the polynomial really is the interpolation polynomial.

c) (J, no hand-in) Plot the interpolation polynomial again, but this time by the use of the python functions `cardinal` and `lagrange`, and make sure that this plot is exactly the same as in the previous point.

By doing the last two points, you confirm that you have found the correct interpolation polynomial, and that you know how to use the functions `cardinal` and `lagrange`.

2 (J) The population of Norway in the period from 1993 to 2018 is, according to SSB,

year	1993	1998	2003	2008	2013	2018
population	4299167	4417599	4552252	4737171	5051275	5295619

Use the interpolation polynomial, through the functions `kardinal` and `lagrange`, to estimate the population in the years 2000 and 2010. Predict the population in 2025 and 2030. Comment on the results.

(The population in 2000 was 4478497, in 2010 it was 4858199).

3 Consider the function $f(x) = x^2 \cos x$.

a) (H) Find the polynomial of degree 3 that interpolates $f(x)$ at four
- equally distributed nodes
- Chebyshev points

in the interval $[-1, 2]$.

b) (H) Find by hand a bound for the maximal interpolation error in that interval in these two cases.

c) (J) Confirm your results numerically:
Plot the function and the interpolation polynomial, measure the maximal interpolation error and compare it with your theoretical result.

d) (J) Repeat the experiments numerically, with $n + 1$ interpolation points, using $n = 5, 10, 15, 20$.
In this case, we are only interested in the measured maximum error and the error bounds.

Hint:

$$\frac{d^n}{dx^n} x^2 \cos(x) = \begin{cases} (-1)^{n/2}\big(x^2 \cos(x) + 2nx \sin(x) - n(n-1)\cos(x)\big) & \text{for } n \text{ even,} \\ (-1)^{(n+1)/2}\big(x^2 \sin(x) - 2nx \cos(x) - n(n-1)\sin(x)\big) & \text{for } n \text{ odd.} \end{cases}$$

4 (Optional, but relevant for the exam!)
The heat equation in 2+1 dimensions is $u_t(x, y, t) = k(u_{xx}(x, y, t) + u_{yy}(x, y, t))$. We consider the case $u_t = 0$, which yields the so-called Laplace equation

$$u_{xx} + u_{yy} = 0.$$

The corresponding problem consists of studying this PDE in some region R of the xy-plane together with a given boundary condition on the boundary curve C of R (you should carefully study the final part of Sect. 12.6 in Kreyszig's textbook). This is a so-called boundary value problem (BVP) of which the Dirichlet, Neumann and Robin problems with their respective Dirichlet, Neumann and Robin boundary conditions are most relevant.

In this exercise we'll consider the Dirichlet problem in a circular region of radius $r < a$ subject to the boundary condition $u(a, \theta) = f(\theta)$, $0 < \theta < 2\pi$, where f is given for $0 \le \theta < 2\pi$ with $f(0) = f(2\pi)$. We state explicitly that the solution $u(r, \theta)$ must be bounded for $r \le a$.

a) As a first step we move from cartesian to polar coordinates. Show that the Laplace equation in polar coordinates reads

$$u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0$$

for $0 < r < a$ and $0 < \theta < 2\pi$. (Hint: use $x = r\cos(\theta)$, $y = r\sin(\theta)$ and the chain rule.)

b) Applying the method of separation of variables: $u(r, \theta) = R(r)\Theta(\theta)$, show that

$$R''\Theta + \frac{1}{r} R'\Theta + \frac{1}{r^2} R\Theta'' = 0,$$

and derive an ODE for $R(r)$ and one for $\Theta(\theta)$.

c) In the second order ODEs for R and Θ derived above, a so-called separation constant λ appears. The periodicity condition implies that $\lambda \in \mathbb{R}$ (you're encouraged to check this). Study the three cases i) $\lambda < 0$; ii) $\lambda = 0$; iii) $\lambda > 0$, and show in detail that only the last case, i.e., $\lambda > 0$, is compatible with the periodicity and boundedness assumption for $u(r, \theta)$.

d) Finally, show that periodicity and boundedness of u imply that the solution

$$u(r, \theta) = c_0 + \sum_{n>0} r^n (\alpha_n \cos(n\theta) + \beta_n \sin(n\theta)),$$

e) Compute the Fourier coefficients $a^n \alpha_n$ and $a^n \beta_n$ by (integrating from 0 to 2π) using the boundary condition $u(a, \theta) = f(\theta)$, for $0 \leq \theta < 2\pi$. Note that the boundary conditions of this problem required both sine and cosine terms in the solution, that is, the full Fourier series for f is required, rather than sine or cosine terms alone.