Norwegian University of Science and Technology
Department of Mathematical Sciences

TMA4135 Calculus 4D
Continuation exam 2016

Solution suggestions ??
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$\boxed{1}$ Newton's divided differences for the data:

$$\begin{array}{c|c|ccc}
0 & 0 & & & \\
 & & \frac{1-0}{1-0} = 1 & & \\
1 & 1 & & \frac{-1/2-1}{2-0} = -\frac{3}{4} & . \\
 & & \frac{1/2-1}{2-1} = -\frac{1}{2} & & \\
2 & 1/2 & & &
\end{array}$$

Thus

$$p(x) = 0 + 1x - \frac{3}{4}(x-1)x = -\frac{3}{4}x^2 + \frac{7}{4}x.$$

$\boxed{2}$ Application of the Laplace-transform to the equation gives us

$$s^2Y - sy(0) - y'(0) + 3sY - 3y(0) + 2Y = \mathcal{L}(tu(t-1)).$$

The right hand side computes to

$$\mathcal{L}(tu(t-1)) = \mathcal{L}(((t-1)+1)u(t-1)) = e^{-s}\mathcal{L}(t+1)$$
$$= e^{-s}\Big(\frac{1}{s^2} + \frac{1}{s}\Big) = e^{-s}\frac{s+1}{s^2}.$$

Inserting this and the initial conditions into the transformed equation, we obtain

$$s^2Y - s + 1 + 3sY - 3 + 2Y = e^{-s}\frac{s+1}{s^2}.$$

Noting that

$$(s^2 + 3s + 2) = (s+2)(s+1)$$

we obtain that

$$(s+2)(s+1)Y = s + 2 + e^{-s}\frac{s+1}{s^2}$$

and thus

$$Y = \frac{1}{s+1} + e^{-s}\frac{1}{s^2(s+2)}.$$

Now

$$\frac{1}{s^2(s+2)} = \frac{1}{2}\Big(\frac{s+2-s}{s^2(s+2)}\Big) = \frac{1}{2}\Big(\frac{1}{s^2} - \frac{1}{s(s+2)}\Big) = \frac{1}{2}\Big(\frac{1}{s^2} - \frac{1}{2}\Big(\frac{s+2-s}{s(s+2)}\Big)\Big) = \frac{1}{2}\Big(\frac{1}{s^2} - \Big(\frac{1}{s} - \frac{1}{s+2}\Big)\frac{1}{2}\Big)$$

Thus

$$y(t) = \mathcal{L}^{-1}\Big(\frac{1}{s+1} + e^{-s}\Big(\frac{1/2}{s^2} - \frac{1/4}{s} + \frac{1/4}{s+2}\Big)\Big)$$
$$= e^{-t} + u(t-1)\Big(\frac{t-1}{2} - \frac{1}{4} + \frac{e^{-2(t-1)}}{4}\Big)$$
$$= e^{-t} + u(t-1)\Big(\frac{1}{2}t - \frac{3}{4} + \frac{1}{4}e^{-2(t-1)}\Big)$$

3 a) The ODE can be written

$$y'''(x) = \frac{x^2 + xy'(x) + y(x)}{1 + x^3}.$$

Define

$$\mathbf{z}(x) = \begin{pmatrix} z_1(x) \\ z_2(x) \\ z_3(x) \end{pmatrix} = \begin{pmatrix} y(x) \\ y'(x) \\ y''(x) \end{pmatrix}.$$

Then

$$\mathbf{z}'(x) = \begin{pmatrix} z_1'(x) \\ z_2'(x) \\ z_3'(x) \end{pmatrix} = \begin{pmatrix} y'(x) \\ y''(x) \\ y'''(x) \end{pmatrix} = \begin{pmatrix} y'(x) \\ y''(x) \\ \frac{x^2+xy'(x)+y(x)}{1+x^3} \end{pmatrix} = \begin{pmatrix} z_2(x) \\ z_3(x) \\ \frac{x^2+xz_2(x)+z_1(x)}{1+x^3} \end{pmatrix}$$

b) We have $\mathbf{z}'(x) = \mathbf{f}(x, \mathbf{z}(x))$ with

$$\mathbf{f}(x, \mathbf{z}) = \begin{pmatrix} z_2 \\ z_3 \\ \frac{x^2+xz_2+z_1}{1+x^3} \end{pmatrix}.$$

~~The backwards Euler~~ method is $\mathbf{z}^{(n+1)} = \mathbf{z}^{(n)} + h\mathbf{f}(x_{n+1}, \mathbf{z}^{(n+1)})$. With $n = 0$ we find

$$\begin{pmatrix} z_1^{(n+1)} \\ z_2^{(n+1)} \\ z_3^{(n+1)} \end{pmatrix} = \begin{pmatrix} z_1^{(n)} \\ z_2^{(n)} \\ z_3^{(n)} \end{pmatrix} + h \begin{pmatrix} z_2^{(n+1)} \\ z_3^{(n+1)} \\ \frac{h^2+hz_2^{(n+1)}+z_1^{(n+1)}}{1+h^2} \end{pmatrix},$$

or in matrix form

$$\begin{pmatrix} 1 & -h & 0 \\ 0 & 1 & -h \\ \frac{-h}{1+h^2} & \frac{-h^2}{1+h^2} & 1 \end{pmatrix} \begin{pmatrix} z_1^{(n+1)} \\ z_2^{(n+1)} \\ z_3^{(n+1)} \end{pmatrix} = \begin{pmatrix} z_1^{(n)} \\ z_2^{(n)} \\ z_3^{(n)} + \frac{h^3}{1+h^2} \end{pmatrix}$$

With $h = 2$, we have no guarantee of convergence of G–S iteration. With $h = 1/2$, the matrix in the system above is strictly diagonally dominant, so the iteration method is guaranteed to converge.

4 a) Standard formulas give (Kreyszig, p. 484)

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx = \frac{1}{L}\int^{L}(L-x)dx = \frac{L}{2},$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx = \frac{2}{L}\int_0^{L}(L-x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$= -\frac{2L}{(n\cancel{\pi})^2}((-1)^n - 1), \quad n \in \mathbb{N},$$

$$b_n = 0, \quad n \in \mathbb{N}.$$

Thus

$$
\begin{aligned}
f(x) &= \frac{L}{2} - \frac{2L}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n-1}{n^2}\cos\left(\frac{n\pi x}{L}\right) \\
&= \frac{L}{2} + \frac{4L}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos\left(\frac{(2n-1)\pi x}{L}\right), \quad x\in[0,L].
\end{aligned}
$$

b) Standard formulas give (Kreyszig, p. 484)

$$
\begin{aligned}
a_0 &= 0, \\
a_n &= 0. \quad n\in\mathbb{N}, \\
b_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx = \frac{2}{L}\int_{0}^{L}(L-x)\sin\left(\frac{n\pi x}{L}\right)dx \\
&= \frac{2L}{n\pi}, \quad n\in\mathbb{N}.
\end{aligned}
$$

Thus

$$
f(x) = \frac{2L}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{n\pi x}{L}\right), \quad x\in(0,L].
$$

c) For $x\in(0,L)$ we have

$$
\begin{aligned}
0 &= \text{Fourier series in (b)} - \text{Fourier series in (a)} \\
&= \frac{2L}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{n\pi x}{L}\right) - \left(\frac{L}{2} + \frac{4L}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos\left(\frac{(2n-1)\pi x}{L}\right)\right)
\end{aligned}
$$

which can easily be rewritten to the given expression. When $x=0$ the Fouries series in (a) sums to L while the Fourier series in (b) sums to zero. Thus the left-hand side equals $-L/2$ for $x=0$. When $x=L$, both Fourier series sum to zero, and thus the identity is still valid for $x=L$.

d) Parseval's formula (Kreyszig, p. 497, replace π by L) gives for the functions in (a)

$$
2a_0^2 + \sum_{n=1}^{\infty}a_n^2 = \frac{1}{L}\int_{-L}^{L} f(x)^2dx
$$

Thus

$$
\frac{1}{2}L^2 + \sum_{n=1}^{\infty}\frac{16L^2}{\pi^4(2n-1)^4} = \frac{2}{3}L^2
$$

or

$$
\sum_{n=1}^{\infty}\frac{1}{(2n-1)^4} = \frac{\pi^4}{16L^2}\left(\frac{2}{3}L^2 - \frac{1}{2}L^2\right) = \frac{\pi^4}{96}.
$$

$\boxed{5}$ We take the Fourier transform (Kreyzsig, p. 527)

$$
\begin{aligned}
\mathcal{F}(f * g) &= \sqrt{2\pi}\hat{f}\,\hat{g} \\
&= \sqrt{2\pi}\,\frac{1}{\sqrt{2}}e^{-\omega^2/4}\big(-\frac{1}{2}\big)\mathcal{F}\big(\frac{d}{dx}e^{-x^2}\big) \\
&= -\frac{\sqrt{2\pi}}{2\sqrt{2}}\,e^{-\omega^2/4}\,i\omega\mathcal{F}(e^{-x^2}) \\
&= -\frac{\sqrt{2\pi}i}{4}\,\omega\, e^{-\omega^2/4}\,e^{-\omega^2/4} \\
&= -\frac{\sqrt{2\pi}i}{4}\omega\, e^{-\omega^2/2} \\
&= \frac{\sqrt{2\pi}i}{4}\frac{d}{d\omega}e^{-\omega^2/2}.
\end{aligned}
$$

Here we also used the formula for the Fourier transform of the derivative (Kreyzsig, p. 526).

Using that $\mathcal{F}^{-1}(h)(x) = \mathcal{F}(h)(-x)$ (Kreyzsig, p. 523f) we see that

$$
\begin{aligned}
f * g &= \mathcal{F}^{-1}\Big(\frac{\sqrt{2\pi}i}{4}\frac{d}{d\omega}e^{-\omega^2/2}\Big) \\
&= \frac{\sqrt{2\pi}i}{4}\mathcal{F}^{-1}\Big(\frac{d}{d\omega}e^{-\omega^2/2}\Big) \\
&= \frac{\sqrt{2\pi}i}{4}\mathcal{F}\Big(\frac{d}{d\omega}e^{-\omega^2/2}\Big)(-x) \\
&= \frac{\sqrt{2\pi}i}{4}i(-x)\frac{1}{\sqrt{2/2}}e^{-(-x)^2/(4/2)} \\
&= \frac{\sqrt{2\pi}}{4}\,x\, e^{-x^2/2}.
\end{aligned}
$$

$\boxed{6}$ **a)** Standard separation of variables gives

$$F'' + kF = 0, \quad G'' - kG = 0$$

for some constant $k \in \mathbb{R}$. Three cases to be considered:
(i) $k = -\lambda^2 < 0$. Using the boundary condition $0 = F'(0) = F'(\pi)$ we see that F is identically zero.
(ii) $k = 0$. Using the boundary condition $0 = F'(0) = F'(\pi)$, we see that F constant is the only solution.
(iii) $k = \lambda^2 > 0$. Using the boundary condition $0 = F'(0) = F'(\pi)$, we see in the standard manner that $\lambda = n \in \mathbb{Z}$ and

$$F(x) = \beta_n \cos(nx), \quad n \in \mathbb{Z},$$

for any constant $\beta_n \in \mathbb{R}$. Since cosinus is an even function it suffices to consider nonnegative integers.
Using this result for the equation for G, we infer that

$$G(y) = A_n e^{ny} + B_n e^{-ny}, \quad n \in \mathbb{N},$$

for any constants $A_n, B_n \in \mathbb{R}$. For $n = 0$ we find $G(y) = A_0 y + B_0$ for constants $A_0, B_0 \in \mathbb{R}$.
Thus the general solution of the form $u = FG$ reads

$$u(x,y) = F(x)G(y) = u_n(x,y) = \begin{cases} A_0 y + B_0, & \text{for } n = 0, \\ \cos(nx)(A_n e^{ny} + B_n e^{-ny}), & \text{for } n \in \mathbb{N}. \end{cases}$$

b) The general solution reads

$$u(x,y) = \sum_{n=0}^{\infty} u_n(x,y) = A_0 y + B_0 + \sum_{n=1}^{\infty} \cos(nx)(A_n e^{ny} + B_n e^{-ny}).$$

The boundary condition at $y = 0$ yields

$$u(x,0) = B_0 + \sum_{n=1}^{\infty} \cos(nx)(A_n + B_n) = 0.$$

By the uniqueness of Fourier series we conclude that $B_0 = 0$ and $A_n = -B_n$ for $n \in \mathbb{N}$. The boundary condition at $y = 0$ yields

$$\begin{aligned} u_y(x, \frac{\pi}{2}) &= A_0 + \sum_{n=1}^{\infty} \cos(nx) n A_n (e^{n\pi/2} + e^{-n\pi/2}) \\ &= (1 + \cos(x))^2 \\ &= 1 + 2\cos(x) + \cos^2(x) \\ &= \frac{3}{2} + 2\cos(x) + \frac{1}{2}\cos(2x). \end{aligned}$$

Again by the uniqueness of Fourier series we conclude that $A_0 = 3/2$, $1A_1(e^{\pi/2} + e^{-\pi/2}) = 2$, and $2A_2(e^{\pi} + e^{-\pi}) = 1/2$, while all the other constants vanish. We may write

$$A_1 = \frac{1}{\cosh(\pi/2)}, \quad A_2 = \frac{1}{8\cosh(\pi)}.$$

Thus

$$u(x,y) = \frac{3}{2} y + \frac{1}{\cosh(\pi/2)} \cos(x)(e^y - e^{-y}) + \frac{1}{8\cosh(\pi)} \cos(2x)(e^{2y} - e^{-2y}).$$