Norges teknisk–naturvitenskapelige universitet
Institutt for matematiske fag

TMA4135 Matematikk 4D
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Løsningsforslag — Øving 7

1 Integralligningen

$$f(x) = e^{-|x|} - 4\int_{-\infty}^{\infty} f(p)e^{-|x-p|}\,\mathrm{d}p$$

kan skrives på formen

$$f(x) = e^{-|x|} - 4f(x) * e^{-|x|} \tag{$*$}$$

Fourier-transformerer så $(*)$, som gir

$$\hat{f}(w) = \sqrt{\frac{2}{\pi}}\frac{1}{1+w^2} - 4\cdot\sqrt{2\pi}\hat{f}(w)\sqrt{\frac{2}{\pi}}\frac{1}{1+w^2} = \sqrt{\frac{2}{\pi}}\frac{1}{1+w^2} - \frac{8\hat{f}(w)}{1+w^2}, \tag{$**$}$$

der vi har benyttet at $(e^{-|x|}) = \sqrt{\frac{2}{\pi}}\frac{1}{1+w^2}$ og at $(f*g) = \sqrt{2\pi}(f)(g)$. Løser så $(**)$ for $\hat{f}(w)$, det vil si

$$\hat{f}(w)\left(1+\frac{8}{1+w^2}\right) = \sqrt{\frac{2}{\pi}}\frac{1}{1+w^2} \qquad \text{slik at} \qquad \hat{f}(w) = \sqrt{\frac{2}{\pi}}\frac{1}{9+w^2}.$$

Altså er

$$\begin{aligned} f(x) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{f}(w)e^{iwx}\,\mathrm{d}w = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\sqrt{\frac{2}{\pi}}\frac{1}{9+w^2}e^{iwx}\,\mathrm{d}w \\ &= \frac{1}{3\sqrt{2\pi}}\int_{-\infty}^{\infty}\underbrace{\sqrt{\frac{2}{\pi}}\frac{3}{9+w^2}}_{(e^{-3|x|})}e^{iwx}\,\mathrm{d}w = \frac{1}{3}e^{-3|x|}. \end{aligned}$$

2 **a)** Vi har at

$$\begin{aligned} \hat{f}(w) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-iwx}\,\mathrm{d}x = \frac{1}{\sqrt{2\pi}}\int_{-1}^{1} e^{-iwx}\,\mathrm{d}x \\ &= \frac{1}{\sqrt{2\pi}}\underbrace{\left[-\frac{1}{iw}e^{-iwx}\right]_{-1}^{1}}_{\frac{2}{w}\sin w} = \sqrt{\frac{2}{\pi}}\frac{\sin w}{w}, \end{aligned}$$

og at

$$\begin{aligned} \hat{g}(w) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(x)e^{-iwx}\,\mathrm{d}x = \frac{1}{\sqrt{2\pi}}\int_{0}^{\infty} e^{-x}e^{-iwx}\,\mathrm{d}x \\ &= \frac{1}{\sqrt{2\pi}}\left[-\frac{1}{1+iw}e^{-(1+iw)x}\right]_0^{\infty} = \frac{1}{\sqrt{2\pi}}\frac{1-iw}{1+w^2}. \end{aligned}$$

b) Ettersom $h(x) = (f * g)(x)$ har vi at

$$\hat{h}(w) = \sqrt{2\pi}\hat{f}(w)\hat{g}(w) = \sqrt{2\pi}\sqrt{\frac{2}{\pi}}\frac{\sin w}{w}\frac{1}{\sqrt{2\pi}}\frac{1-iw}{1+w^2} = \sqrt{\frac{2}{\pi}}\frac{(1-iw)\sin w}{w(1+w^2)},$$

slik at

$$h(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\sqrt{\frac{2}{\pi}}\frac{(1-iw)\sin w}{w(1+w^2)}e^{iwx}\ \mathrm{d}w = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{(1-iw)\sin w}{w(1+w^2)}e^{iwx}\ \mathrm{d}w.$$

Fra definisjonen til $h(x)$ får vi at

$$h(0) = \int_{-\infty}^{\infty} f(p)g(0-p)\ \mathrm{d}p = \int_{-\infty}^{\infty} f(p)g(-p)\ \mathrm{d}p = \int_{-1}^{0} e^p\ \mathrm{d}p = 1 - e^{-1}.$$

Vi observerer at

$$h(0) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{(1-iw)\sin w}{w(1+w^2)}\ \mathrm{d}w = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sin w}{w(1+w^2)}\ \mathrm{d}w - \frac{i}{\pi}\int_{-\infty}^{\infty}\frac{\sin w}{1+w^2}\ \mathrm{d}w = 1-e^{-1}.$$

Ved å sammenligne realdelene på begge sider av (det siste) likhetstegnet, ser vi at

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sin w}{w(1+w^2)}\ \mathrm{d}w = 1 - e^{-1}$$

det vil si

$$\int_{-\infty}^{\infty}\frac{\sin w}{w(1+w^2)}\ \mathrm{d}w = \pi\big(1 - e^{-1}\big).$$

3 Fiksér x, slik at

$$u_y = -2yu,$$

kan tenkes på som en separabel ordinær differensialligning (med y som variabel) der

$$\frac{\mathrm{d}u}{u} = -2y\ \mathrm{d}y.$$

Integerer opp begge sider av likheten og får at

$$\ln u = -y^2 + k(x),$$

det vil si

$$u(x,y) = c(x)e^{-y^2},$$

der $c(x) = e^{k(x)}$.

4 Fiksér y, slik at

$$u_{xx} - 4y^2u = 0,$$

kan tenkes på som en separabel ordinær differensialligning (med x som variabel) der den karakteristiske ligningen

$$\lambda^2 - 4y^2 = 0 \qquad \text{har løsning} \qquad \lambda = \pm 2y.$$

Altså har vi at

$$u(x,y) = A(y)e^{2xy} + B(y)e^{-2xy}.$$

5 Fra Kreyszig, avsnitt 12.3 (9th og 10th ed), har vi at den generelle løsningen til den éndimensjonale bølgeligningen (med $c^2 = L = 1$ og $u_t(x,0) = 0$) er gitt ved

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} B_n \cos n\pi t \sin n\pi x,$$

der

$$B_n = 2\int_0^1 f(x) \sin n\pi x \, \mathrm{d}x \qquad n = 1, 2, 3, \ldots$$

I vårt tilfelle er

$$f(x) = \begin{cases} x & \text{for } 0 \le x \le \frac{1}{4}, \\ \frac{1}{4} & \text{for } \frac{1}{4} < x < \frac{3}{4}, \\ 1 - x & \text{for } \frac{3}{4} \le x \le 1, \end{cases}$$

slik at

$$\begin{aligned}
B_n &= 2\int_0^1 f(x) \sin n\pi x \, \mathrm{d}x \\
&= 2\left(\int_0^{\frac{1}{4}} x \sin n\pi x \, \mathrm{d}x + \int_{\frac{1}{4}}^{\frac{3}{4}} \frac{1}{4} \sin n\pi x \, \mathrm{d}x + \int_{\frac{3}{4}}^{1} (1-x) \sin n\pi x \, \mathrm{d}x \right) \\
&= 2\left(\left[-\frac{x}{n\pi} \cos n\pi x \right]_0^{\frac{1}{4}} + \frac{1}{n\pi} \int_0^{\frac{1}{4}} \cos n\pi x \, \mathrm{d}x + \left[-\frac{1}{4n\pi} \cos n\pi x \right]_{\frac{1}{4}}^{\frac{3}{4}} + \left[-\frac{1}{n\pi} \cos n\pi x \right]_{\frac{3}{4}}^{1} \right. \\
&\qquad \left. - \int_{\frac{3}{4}}^{1} x \sin n\pi x \, \mathrm{d}x \right) \qquad \text{(delvis integrasjon)} \\
&= 2\left(\frac{3}{4n\pi} \cos \frac{3n\pi}{4} - \frac{1}{n\pi} (-1)^n + \left[\frac{1}{n^2\pi^2} \sin n\pi x \right]_0^{\frac{1}{4}} + \left[\frac{x}{n\pi} \cos n\pi x \right]_{\frac{3}{4}}^{1} \right. \\
&\qquad \left. - \frac{1}{n\pi} \int_{\frac{3}{4}}^{1} \cos n\pi x \, \mathrm{d}x \right) \qquad \text{(delvis integrasjon)} \\
&= 2\left(\frac{1}{n^2\pi^2} \sin \frac{n\pi}{4} + \left[-\frac{1}{n^2\pi^2} \sin n\pi x \right]_{\frac{3}{4}}^{1} \right) \\
&= \frac{2}{n^2\pi^2} \left(\sin \frac{n\pi}{4} + \sin \frac{3n\pi}{4} \right).
\end{aligned}$$

Altså har vi at

$$\begin{aligned}
u(x,t) &= \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} B_n \cos n\pi t \sin n\pi x \\
&= \frac{2\sqrt{2}}{\pi^2} \left(\cos \pi t \sin \pi x + \frac{1}{9} \cos 3\pi t \sin 3\pi x - \frac{1}{25} \cos 5\pi t \sin 5\pi x - + \cdots \right).
\end{aligned}$$

6 Fra Kreyszig, avsnitt 12.3 (9th og 10th ed), har vi at den generelle løsningen til den éndimensjonale bølgeligningen (med $c^2 = L = 1$ og $u_t(x,0) = 0$) er gitt ved

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} B_n \cos n\pi t \sin n\pi x,$$

der

$$B_n = 2\int_0^1 f(x)\sin n\pi x \;\mathrm{d}x \qquad n = 1, 2, 3, \dots$$

I vårt tilfelle er

$$f(x) = \begin{cases} x & \text{for } 0 \le x \le \frac{1}{4}, \\ \frac{1}{2} - x & \text{for } \frac{1}{4} < x < \frac{3}{4}, \\ x - 1 & \text{for } \frac{3}{4} < x < 1, \end{cases}$$

slik at

$$\begin{aligned}
B_n &= 2\int_0^1 f(x)\sin n\pi x \;\mathrm{d}x \\
&= 2\left(\int_0^{\frac{1}{4}} x\sin n\pi x \;\mathrm{d}x + \int_{\frac{1}{2}}^{\frac{3}{4}} \left[\frac{1}{2} - x\right]\sin n\pi x \;\mathrm{d}x + \int_{\frac{3}{4}}^{1} (x-1)\sin n\pi x \;\mathrm{d}x\right) \\
&= 2\left(\left[-\frac{x}{n\pi}\cos n\pi x\right]_0^{\frac{1}{4}} + \frac{1}{n\pi}\int_0^{\frac{1}{4}} \cos n\pi x \;\mathrm{d}x + \left[-\frac{1}{2n\pi}\cos n\pi x\right]_{\frac{1}{4}}^{\frac{3}{4}} - \int_{\frac{1}{4}}^{\frac{3}{4}} x\sin n\pi x \;\mathrm{d}x \right. \\
&\qquad \left. + \int_{\frac{3}{4}}^{1} x\sin n\pi x \;\mathrm{d}x + \left[\frac{1}{n\pi}\cos n\pi x\right]_{\frac{3}{4}}^{1}\right) \qquad \text{(delvis integrasjon)} \\
&= 2\left(\frac{1}{4n\pi}\cos\frac{n\pi}{4} - \frac{3}{2n\pi}\cos\frac{3n\pi}{4} + \frac{1}{n\pi}(-1)^n + \left[\frac{1}{n^2\pi^2}\sin n\pi x\right]_0^{\frac{1}{4}} + \left[\frac{x}{n\pi}\cos n\pi x\right]_{\frac{1}{4}}^{\frac{3}{4}} \right. \\
&\qquad \left. - \frac{1}{n\pi}\int_{\frac{1}{4}}^{\frac{3}{4}} \cos n\pi x \;\mathrm{d}x + \left[-\frac{x}{n\pi}\cos n\pi x\right]_{\frac{3}{4}}^{1} + \frac{1}{n\pi}\int_{\frac{3}{4}}^{1} \cos n\pi x \;\mathrm{d}x\right) \\
&\qquad \text{(delvis integrasjon)} \\
&= 2\left(\frac{1}{n^2\pi^2}\sin\frac{n\pi}{4} - \left[\frac{1}{n^2\pi^2}\sin n\pi x\right]_{\frac{1}{4}}^{\frac{3}{4}} + \left[\frac{1}{n^2\pi^2}\sin n\pi x\right]_{\frac{3}{4}}^{1}\right) \\
&= \frac{4}{n^2\pi^2}\left(\sin\frac{n\pi}{4} - \sin\frac{3n\pi}{4}\right).
\end{aligned}$$

Altså har vi at

$$\begin{aligned}
u(x,t) &= \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} B_n \cos n\pi t \sin n\pi x \\
&= \frac{8}{\pi^2}\left(\frac{1}{4}\sin 2\pi x\cos 2\pi t - \frac{1}{36}\sin 6\pi x\cos 6\pi t + \frac{1}{100}\sin 10\pi x\cos 10\pi t - + \cdots\right).
\end{aligned}$$