Norges teknisk–naturvitenskapelige universitet
Institutt for matematiske fag

Løsningsforslag — Øving 3

1 Ved å benytte formel (21) fra tabellen over Laplace-transformerte (Section 6.9) i boken sammen med forskyvning langs s-aksen (First Shifting Theorem, s-shifting) får vi at

$$\mathscr{L}(f) = \frac{2(s+k)}{\big((s+k)^2+1\big)^2}.$$

2 Forskyvning langs s-aksen (First Shifting Theorem, s-shifting) samt at $\mathscr{L}(t) = \frac{1}{s^2}$ gir at

$$f(t) = \mathscr{L}^{-1}\left(\frac{6}{(s+1)^2}\right) = 6te^{-t}.$$

3 Funksjonen er gitt ved

$$f(x) = \begin{cases} 0 & \text{for } -\pi < x < 0, \\ 1 & \text{for } 0 < x < \pi, \end{cases}$$

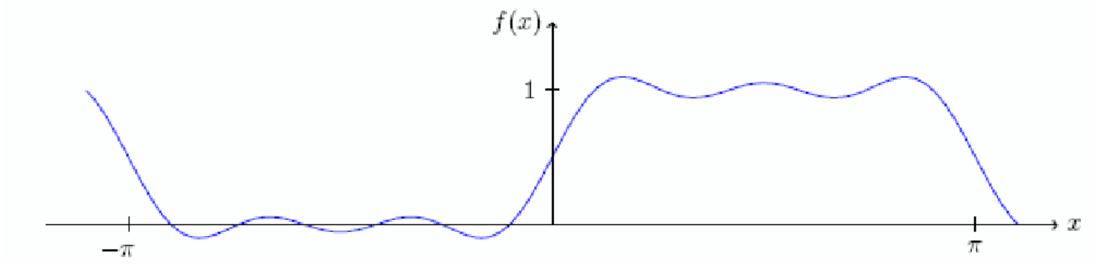
samt at $f(x+2\pi) = f(x)$ for alle x. Dermed får vi at Fourier-koeffisientene til $f(x)$ er gitt ved

$$\begin{aligned} a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\ \mathrm{d}x = \frac{1}{2}, \\ a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\ \mathrm{d}x = \frac{1}{\pi}\int_0^{\pi}\cos nx\ \mathrm{d}x = 0, \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\ \mathrm{d}x = \frac{1}{\pi}\int_0^{\pi}\sin nx\ \mathrm{d}x = \frac{1}{n\pi}(1-\cos n\pi). \end{aligned}$$

Vi merker oss at for alle naturlige tall m gjelder det at $b_{2m} = \frac{1}{2m\pi}(1-\cos 2m\pi) = 0$, og videre at $b_{2m+1} = \frac{1}{(2m+1)\pi}\big(1-\cos(2m+1)\pi\big) = \frac{2}{(2m+1)\pi}$. Altså er Fourier-rekken til $f(x)$ gitt ved

$$\begin{aligned} f(x) &= a_0 + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx) = \frac{1}{2} + \sum_{m=0}^{\infty}\frac{2}{(2m+1)\pi}\sin(2m+1)x \\ &= \frac{1}{2} + \frac{2}{\pi}\sin x + \frac{2}{3\pi}\sin 3x + \cdots. \end{aligned}$$

Grafen til $\frac{1}{2} + \frac{2}{\pi}\sin x + \frac{2}{3\pi}\sin 3x + \frac{2}{5\pi}\sin 5x$ er gitt under.



4 Funksjonen er gitt ved

$$f(x) = \begin{cases} -\frac{1}{2}\pi & \text{for } -\pi < x < \frac{1}{2}\pi, \\ x & \text{for } -\frac{1}{2}\pi < x < \frac{1}{2}\pi, \\ \frac{1}{2}\pi & \text{for } \frac{1}{2}\pi < x < \pi, \end{cases}$$

samt at $f(x+2\pi) = f(x)$. Legg merke til at $f(-x) = -f(x)$ for alle x, så med andre ord er $f(x)$ en odde funksjon. Dermed er Fourier-koeffisientene til $f(x)$ gitt ved

$$\begin{aligned} a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\ \mathrm{d}x = 0, \\ a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\ \mathrm{d}x = 0, \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\ \mathrm{d}x \\ &= \frac{1}{\pi}\left\{\int_{-\pi}^{-\frac{1}{2}\pi} -\frac{1}{2}\pi\sin nx\ \mathrm{d}x + \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} x\sin nx \mathrm{d}x + \int_{\frac{1}{2}\pi}^{\pi} \frac{1}{2}\pi\sin nx\ \mathrm{d}x\right\} \\ &= I_1 + I_2 + I_3, \end{aligned}$$

der $I_1 = -\frac{1}{2}\int_{-\pi}^{-\frac{1}{2}\pi}\sin nx\ \mathrm{d}x$, $I_2 = \frac{1}{\pi}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} x\sin nx\ \mathrm{d}x$ og $I_3 = \frac{1}{2}\int_{\frac{1}{2}\pi}^{\pi}\sin nx\ \mathrm{d}x$. Ved å gjøre et substitusjonsbytte, for eksempel $x = -t$, ser vi at $I_1 = I_3$. Da vi har at

$$\begin{aligned} I_2 &= \frac{1}{\pi}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} x\sin nx\ \mathrm{d}x = \frac{1}{\pi}\left\{\left[-\frac{1}{n}x\cos nx\right]_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} + \frac{1}{n}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\cos nx\ \mathrm{d}x\right\} \\ &= \frac{1}{\pi}\left\{-\frac{\pi}{n}\cos\frac{n\pi}{2} + \frac{1}{n^2}\left[\sin nx\right]_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\right\} = \frac{2}{n^2\pi}\sin\frac{n\pi}{2} - \frac{1}{n}\cos\frac{n\pi}{2}, \\ I_3 &= \frac{1}{2}\int_{\frac{1}{2}\pi}^{\pi}\sin nx\ \mathrm{d}x = -\frac{1}{2n}\left[\cos nx\right]_{\frac{1}{2}\pi}^{\pi} = -\frac{1}{2n}\left(\cos n\pi - \cos\frac{n\pi}{2}\right), \end{aligned}$$

får vi at

$$b_n = I_2 + 2I_3 = \frac{2}{n^2\pi}\sin\frac{n\pi}{2} - \frac{1}{n}\cos\frac{n\pi}{2} + \frac{1}{n}\left(\cos\frac{n\pi}{2} - \cos n\pi\right) = \frac{2}{n^2\pi}\sin\frac{n\pi}{2} - \frac{1}{n}\cos n\pi.$$

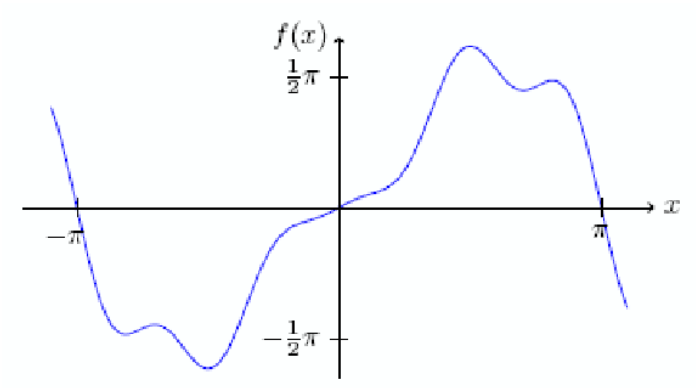
For alle naturlige tall m har vi da at

$$b_{4m-3} = \frac{2}{(4m-3)^2\pi} + \frac{1}{4m-3}, \quad b_{2m} = -\frac{1}{2m}, \quad \text{samt at} \quad b_{4m-1} = -\frac{2}{(4m-1)^2\pi} + \frac{1}{4m-1}.$$

Fourier-rekken til $f(x)$ er da gitt ved

$$\begin{aligned} f(x) &= a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \\ &= \sum_{m=1}^{\infty}(b_{4m-3}\sin(4m-3)x + b_{2m}\sin 2mx + b_{4m-1}\sin(4m-1)x \\ &= \left(\frac{2}{\pi}+1\right)\sin x - \frac{1}{2}\sin 2x + \left(\frac{1}{3}-\frac{2}{9\pi}\right)\sin 3x - \frac{1}{4}\sin 4x + \left(\frac{2}{25\pi}+\frac{1}{5}\right)\sin 5x + \cdots . \end{aligned}$$

Grafen til $\left(\frac{2}{\pi}+1\right)\sin x - \frac{1}{2}\sin 2x + \left(\frac{1}{3}-\frac{2}{9\pi}\right)\sin 3x - \frac{1}{4}\sin 4x + \left(\frac{2}{25\pi}+\frac{1}{5}\right)\sin 5x$ er gitt under.



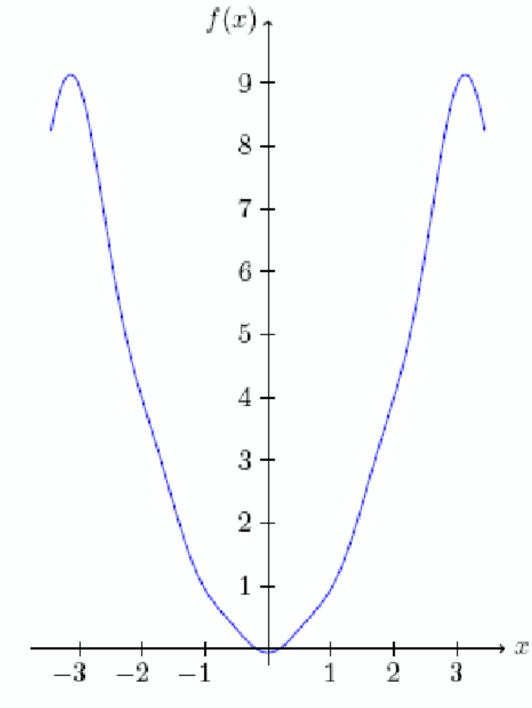
5 Fourier-koeffisientene til den jevne funksjonen $f(x) = x^2$ for $-\pi < x < \pi$, er gitt ved

$$\begin{aligned} a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\ \mathrm{d}x = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2\ \mathrm{d}x = \frac{\pi^2}{3}, \\ a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\ \mathrm{d}x = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cos nx\ \mathrm{d}x \\ &= \frac{1}{\pi}\left\{\frac{1}{n}\left[x^2\sin nx\right]_{-\pi}^{\pi} - \frac{2}{n}\int_{-\pi}^{\pi} x\sin nx\ \mathrm{d}x\right\} = \frac{1}{\pi}\left\{0 + \frac{2}{n^2}\left[x\cos nx\right]_{-\pi}^{\pi} - \frac{2}{n^2}\int_{-\pi}^{\pi}\cos nx\ \mathrm{d}x\right\} \\ &= \frac{1}{\pi}\left\{\frac{4\pi}{n^2}\cos n\pi + 0\right\} = \frac{4}{n^2}(-1)^n, \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\ \mathrm{d}x = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\sin nx\ \mathrm{d}x = 0, \end{aligned}$$

For alle naturlige tall m har vi at $a_{2m-1} = -\frac{4}{(2m-1)^2}$ og at $a_{2m} = \frac{1}{m^2}$. Fourier-rekken til $f(x)$ er dermed gitt ved

$$\begin{aligned} f(x) &= a_0 + \sum_{n=1}^{\infty}(a_n\cos nx + b_n \sin nx) = \frac{\pi^2}{3} + \sum_{m=1}^{\infty}(a_{2m-1}\cos(2m-1)x + a_{2m}\cos 2mx) \\ &= \frac{\pi^2}{3} - 4\cos x + \cos 2x - \frac{4}{9}\cos 3x + \frac{1}{4}\cos 4x - \frac{4}{25}\cos 5x + \cdots . \end{aligned}$$

Grafen til $\frac{\pi^2}{3} - 4\cos x + \cos 2x - \frac{4}{9}\cos 3x + \frac{1}{4}\cos 4x - \frac{4}{25}\cos 5x$ er gitt under.



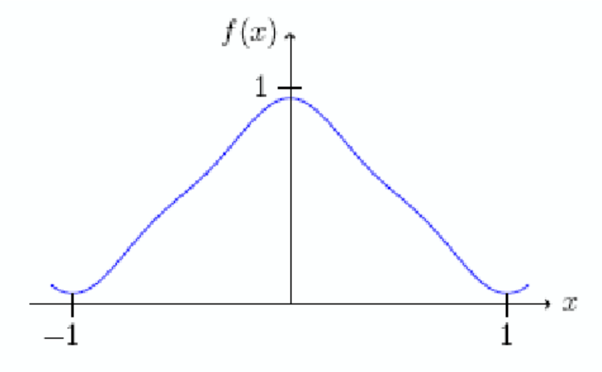
6 Vi har fått oppgitt at funksjonen $f(x)$ har periode $p = 2L = 2$, det vil si, $L = 1$. Fra definisjonen til $f(x)$ ser vi at $f(-x) = f(x)$, det vil at funksjonen er jevn. Fourier-koeffisientene er da gitt ved

$$
\begin{aligned}
a_0 &= \frac{1}{2L}\int_{-L}^{L} f(x)\ \mathrm{d}x = \frac{1}{2}\int_{-1}^{1} f(x)\ \mathrm{d}x = \int_0^1 (1-x)\ \mathrm{d}x = \frac{1}{2},\\
a_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\ \mathrm{d}x = \int_{-1}^{1} f(x)\cos n\pi x\ \mathrm{d}x = 2\int_0^1 (1-x)\cos n\pi x\ \mathrm{d}x\\
&= \frac{2}{n\pi}\left\{\big[(1-x)\sin n\pi x\big]_0^1 + \int_0^1 \sin n\pi x\ \mathrm{d}x\right\} = \frac{2}{n\pi}\left\{0 - \frac{1}{n\pi}\big[\cos n\pi x\big]_0^1\right\},\\
&= \frac{2}{n^2\pi^2}\big(1-(-1)^n\big),\\
b_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\ \mathrm{d}x = \int_{-1}^{1} f(x)\sin n\pi x\ \mathrm{d}x = 0.
\end{aligned}
$$

For alle naturlige tall m gjelder det da at $a_{2m-1} = \frac{4}{(2m-1)^2\pi^2}$ og at $a_{2m} = 0$. Fourier-rekken til $f(x)$ er da gitt ved

$$
\begin{aligned}
f(x) &= a_0 + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx) = \frac{1}{2} + \sum_{m=1}^{\infty}\frac{4}{(2m-1)^2\pi^2}\cos(2m-1)x\\
&= \frac{1}{2} + \frac{4}{\pi^2}\cos\pi x + \frac{4}{9\pi^2}\cos 3\pi x + \cdots .
\end{aligned}
$$

Grafen til summen av Fourier-rekkens tre første ledd er gitt under.


f(x)
1
x
−1
1