1

$$\mathscr{L}(t^2-2t)=\int_0^\infty (t^2-2t)e^{-st}\mathrm{d}t=\int_0^\infty t^2e^{-st}\mathrm{d}t-\int_0^\infty 2te^{-st}\mathrm{d}t=\frac{2}{s^3}-\frac{2}{s^2}$$

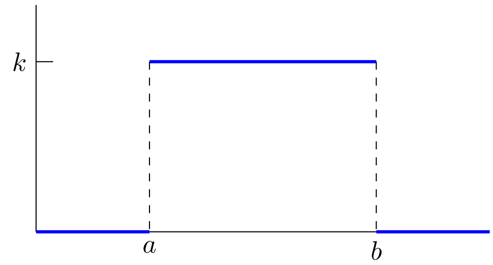
2 Ettersom

$$\begin{aligned}\sin^2 4t&=\left(\frac{1}{2i}(e^{4it}-e^{-4it})\right)^2=-\frac{1}{4}(e^{8it}-2+e^{-8it})\\&=\frac{1}{2}\left(1-\frac{1}{2}(e^{8it}+e^{-8it})\right)=\frac{1}{2}(1-\cos 8t),\end{aligned}$$

får vi at

$$\mathscr{L}(\sin^2 4t)=\mathscr{L}\left(\frac{1}{2}-\frac{1}{2}\cos 8t\right)=\frac{1}{2}\frac{1}{s}-\frac{1}{2}\frac{s}{s^2+8^2}=\frac{1}{2s}-\frac{s}{2(s^2+64)}=\frac{32}{s(s^2+64)}.$$

3 Funksjonen beskrevet ved grafen



kan skrives som

$$f(t)=\begin{cases}k & \text{for } a<t<b,\\ 0 & \text{ellers.}\end{cases}$$

Dette gir at

$$\mathscr{L}(f)=\int_0^\infty f(t)e^{-st}\mathrm{d}t=\int_a^b ke^{-st}\mathrm{d}t=\left[-\frac{k}{s}e^{-st}\right]_a^b=\frac{k}{s}(e^{-sa}-e^{-sb}).$$

4 Ettersom $\mathscr{L}^{-1}\left(\frac{n!}{s^{n+1}}\right)=t^n$ for $n=0,1,2,\ldots$ får vi at

$$\begin{aligned}f(t)=\mathscr{L}^{-1}(F)&=\mathscr{L}^{-1}\left(\frac{s^4-3s^2+12}{s^5}\right)=\mathscr{L}^{-1}\left(\frac{1}{s}-\frac{3}{s^3}+\frac{12}{s^5}\right)=\mathscr{L}^{-1}\left(\frac{1}{s}-\frac{3}{2}\frac{2!}{s^3}+\frac{1}{2}\frac{4!}{s^5}\right)\\&=\mathscr{L}^{-1}\left(\frac{1}{s}\right)-\frac{3}{2}\mathscr{L}^{-1}\left(\frac{2!}{s^3}\right)+\frac{1}{2}\mathscr{L}^{-1}\left(\frac{4!}{s^5}\right)=1-\frac{3}{2}t^2+\frac{1}{2}t^4.\end{aligned}$$

5

$$\begin{aligned}\mathscr{L}^{-1}\left(\frac{1}{s^2+5}-\frac{1}{s+5}\right)&=\mathscr{L}^{-1}\left(\frac{1}{\sqrt{5}}\frac{\sqrt{5}}{s^2+(\sqrt{5})^2}-\frac{1}{s+5}\right)\\&=\mathscr{L}^{-1}\left(\frac{1}{\sqrt{5}}\frac{\sqrt{5}}{s^2+(\sqrt{5})^2}\right)-\mathscr{L}^{-1}\left(\frac{1}{s+5}\right)=\frac{1}{\sqrt{5}}\sin\sqrt{5}t-e^{-5t}\end{aligned}$$

6 Ettersom

$$\begin{aligned}F(s)&=\mathscr{L}(\cos 2t)=\frac{s}{s^2+4},\\G(s)&=\mathscr{L}(\sin 2t)=\frac{2}{s^2+4},\end{aligned}$$

gir linearitet og forskyning langs s-aksen (første forskyvningsteorem, side 224 i boken) at

$$\begin{aligned}\mathscr{L}^{-1}\left(\frac{s-6}{(s-1)^2+4}\right) &= \mathscr{L}^{-1}\left(\frac{s-1}{(s-1)^2+4} - \frac{5}{(s-1)^2+4}\right) \\ &= \mathscr{L}^{-1}\big(F(s-1)\big) - \frac{5}{2}\mathscr{L}^{-1}\big(G(s-1)\big) = e^t\cos 2t - \frac{5}{2}e^t\sin 2t.\end{aligned}$$

7 Vi er gitt initialverdiproblemet

$$y' + \frac{1}{2}y = 17\sin 2t, \qquad y(0) = -1. \tag{$*$}$$

Ved å anvende Laplace-transformasjonen på alle leddene i $(*)$ får vi at

$$\begin{aligned} sY - y(0) + \frac{1}{2}Y &= 17\cdot\frac{2}{s^2+4} \\ \left(s+\frac{1}{2}\right)Y - (-1) &= 17\cdot\frac{2}{s^2+4},\end{aligned}$$

der $Y = Y(s) = \mathscr{L}(y)$. Løser så ut for Y, det vil si

$$Y = \frac{34}{(s^2+4)(s+\frac{1}{2})} - \frac{1}{s+\frac{1}{2}}.$$

Delbrøkoppspaltning gir at

$$\frac{34}{(s^2+4)(s+\frac{1}{2})} = -\frac{8s}{s^2+4} + \frac{4}{s^2+4} + \frac{8}{s+\frac{1}{2}},$$

som igjen gir at

$$Y = -\frac{8s}{s^2+4} + \frac{4}{s^2+4} + \frac{7}{s+\frac{1}{2}}.$$

Fra tabell 6.1, side 224 i boken, får vi da at

$$y(t) = \mathscr{L}^{-1}(Y) = -8\cos 2t + 2\sin 2t + 7e^{-\frac{t}{2}}.$$

8 Vi er gitt initialverdiproblemet

$$y'' + 7y' + 12y = 21e^{3t}, \qquad y(0) = 3{,}5, \qquad y'(0) = -10. \tag{$**$}$$

Ved å anvende Laplace-transformasjonen på alle leddene i $(**)$ får vi at

$$s^2Y - 3{,}5s + 10 + 7sY - 24{,}5 + 12Y = \frac{21}{s-3},$$

der $Y = Y(s) = \mathscr{L}(y)$. Løser så ut for Y, det vil si

$$\begin{aligned} Y &= \frac{21}{(s^2+7s+12)(s-3)} + \frac{3{,}5s}{s^2+7s+12} + \frac{14{,}5}{s^2+7s+12} \\ &= \frac{21}{(s+4)(s+3)(s-3)} + \frac{3{,}5s}{(s+4)(s+3)} + \frac{14{,}5}{(s+4)(s+3)}.\end{aligned}$$

Delbrøkoppspaltning gir så at

$$Y = \frac{2{,}5}{s+4} + \frac{0{,}5}{s+3} + \frac{0{,}5}{s-3}.$$

Altså får vi at

$$y(t) = \mathscr{L}^{-1}(Y) = 2{,}5e^{-4t} + 0{,}5e^{-3t} + 0{,}5e^{3t} = 2{,}5e^{-4t} + \cosh 3t.$$

9 Teorem 3, side 229 i boken, forteller oss at dersom $F(S)$ er den Laplace-transformerte til $f(t)$, det vil si $F(s) = \mathscr{L}(f)$, så har vi at

$$\int_0^t f(\tau)\mathrm{d}\tau = \mathscr{L}^{-1}\left(\frac{1}{s}F(S)\right).$$

I vårt tilfelle har vi at

$$\frac{1}{s^4+\pi^2 s^2} = \frac{1}{s^2}\frac{1}{s^2+\pi^2} = \frac{1}{s^2}\frac{1}{\pi}\frac{\pi}{s^2+\pi^2},$$

slik at ved å benytte teorem 3 to ganger, samt bruke at $\sin \pi t = \mathscr{L}^{-1}\left(\frac{\pi}{s^2+\pi^2}\right)$, får vi at

$$\begin{aligned} f(t) &= \int_0^t \int_0^\tau \frac{1}{\pi}\sin\pi\lambda \ \mathrm{d}\lambda \ \mathrm{d}\tau = \frac{1}{\pi}\int_0^t \left[-\frac{1}{\pi}\cos\pi\lambda\right]_0^\tau \mathrm{d}\tau \\ &= \frac{1}{\pi^2}\int_0^t (1-\cos\pi\tau)\mathrm{d}\tau = \frac{1}{\pi^2}\left[\tau - \frac{1}{\pi}\sin\pi\tau\right]_0^t \\ &= \frac{t}{\pi^2} - \frac{1}{\pi^3}\sin\pi t. \end{aligned}$$