Department of Mathematical Sciences

Problem 1 Laplace transform [10 pts]

Consider the second-order differential equation:

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y(t) = u(t),$$

where $y(t)$ is the system's output and $u(t)$ is the input.

a) Find the Laplace transform of $y(t)$ in terms of the Laplace transform of the input $U(s)$ and any necessary constants. Show all your work and provide a step-by-step solution.

b) Determine the time-domain solution $y(t)$ for the given system when the input $u(t)$ is the unit step function, i.e., $u(t) = 1$ for $t \geq 0$ and $u(t) = 0$ for $t < 0$. Find the expression for $y(t)$.

Solution

a) Let us find the Laplace transform of $y(t)$ from

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y(t) = u(t).$$

Taking the Laplace transform of both sides of the equation, we use the linearity property of the Laplace transform:

$$\mathcal{L}(y'') + 3\mathcal{L}(y') + 2\mathcal{L}(y) = \mathcal{L}(u).$$

The Laplace transform of the derivatives y'' and y' can be expressed using the Laplace transform properties:

$$s^2Y(s) - sy(0) - y'(0) + 3sY(s) - 3y(0) + 2Y(s) = U(s).$$

Now, let us collect terms with $Y(s)$ on one side:

$$Y(s)(s^2 + 3s + 2) = sy(0) + y'(0) + 3y(0) + U(s).$$

$s^2 + 3s + 2$ factors to $(s + 1)(s + 2)$. Now, we can write this as:

$$Y(s) = \frac{sy(0) + y'(0) + 3y(0) + U(s)}{(s + 1)(s + 2)}.$$

So, the Laplace transform of $y(t)$ is given by this expression. The Laplace transform of the output $Y(s)$ depends on the Laplace transform of the input $U(s)$, as well as the initial conditions.

b) When the input $u(t)$ is the unit step function, its Laplace transform is

$$U(s) = \frac{1}{s}.$$

Then $Y(s)$ becomes

$$\begin{aligned} Y(s) &= \frac{sy(0) + y'(0) + 3y(0) + U(s)}{(s+1)(s+2)} \\ &= \frac{sy(0)}{(s+1)(s+2)} + \frac{y'(0) + 3y(0)}{(s+1)(s+2)} + \frac{1}{s(s+1)(s+2)}. \end{aligned}$$

To find the inverse Laplace transform of $\frac{1}{s(s+1)(s+2)}$, we find A, B and C in

$$\frac{1}{s(s+1)(s+2)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2},$$

and we obtain

$$A = \frac{1}{2}, \quad B = -1, \quad C = \frac{1}{2}.$$

We have

$$\begin{aligned} \mathcal{L}^{-1}\left(\frac{sy(0)}{(s+1)(s+2)}\right) &= y(0)(-e^{-t} + 2e^{-2t}) \qquad \text{(see formula sheet)} \\ \mathcal{L}^{-1}\left(\frac{y'(0) + 3y(0)}{(s+1)(s+2)}\right) &= (y'(0) + 3y(0))\left(e^{-t} - e^{-2t}\right) \qquad \text{(see formula sheet)} \\ \mathcal{L}^{-1}\left(\frac{1}{s(s+1)(s+2)}\right) &= \mathcal{L}^{-1}\left(\frac{1}{2s} - \frac{1}{s+1} + \frac{1}{2(s+2)}\right) = \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t}. \end{aligned}$$

Summing up these terms, we obtain the time-domain solution $y(t)$:

$$\begin{aligned} y(t) &= y(0)(-e^{-t} + 2e^{-2t}) + (y'(0) + 3y(0))\left(e^{-t} - e^{-2t}\right) + \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t} \\ &= e^{-t}\left(-y(0) + y'(0) + 3y(0) - 1\right) + e^{-2t}\left(2y(0) - y'(0) - 3y(0) + \frac{1}{2}\right) + \frac{1}{2}. \end{aligned}$$

Grading manual

a) (5p)

b) (5p)

Problem 2 Fourier series [10 pts] (4N)

An even function f of period 2π is approximated by the Nth partial sum of a Fourier cosine series. The error in the approximation is measured by the mean-square deviation

$$E_N = \int_{-\pi}^{\pi} \left(f(x) - a_0 - \sum_{n=1}^{N} a_n \cos(nx) \right)^2 dx.$$

By differentiating E_N with respect to the coefficients a_n, find the values of a_n $(n = 0, 1, 2, \ldots)$ that minimize E_N.

Remark: It is not necessary for you to compute the matrix of second partial derivatives here. Finding the values of a_n $(n = 0, 1, 2, \ldots)$ that solve $\frac{\partial E_N}{\partial a_n} = 0$ is enough.

Solution For $k = 1, 2, 3, \ldots$, we get:

$$\begin{aligned}
\frac{\partial E_N}{\partial a_k} &= -2 \int_{-\pi}^{\pi} \left(f(x) - a_0 - \sum_{n=1}^{N} a_n \cos(nx) \right) \cdot \cos(kx) dx \\
&= -2 \int_{-\pi}^{\pi} f(x) \cdot \cos(kx) dx + 2a_0 \underbrace{\int_{-\pi}^{\pi} \cos(kx) dx}_{=0} + 2 \int_{-\pi}^{\pi} \sum_{n=1}^{N} a_n \cos(nx) \cdot \cos(kx) dx \\
&= -2 \int_{-\pi}^{\pi} f(x) \cdot \cos(kx) dx + \underbrace{2 \sum_{n=1}^{N} a_n \int_{-\pi}^{\pi} \cos(nx) \cdot \cos(kx) dx}_{=2 \int_{\pi}^{\pi} \cos(kx) \cdot \cos(kx) dx = \pi} \\
&\qquad \text{(orthogonality of the trigonometric system)} \\
&= -2 \int_{-\pi}^{\pi} f(x) \cdot \cos(kx) dx + 2\pi a_k.
\end{aligned}$$

Setting $\frac{\partial E_N}{\partial a_k}$ to 0 gives

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(kx) dx.$$

For a_0, we obtain:

$$\begin{aligned}
\frac{\partial E_N}{\partial a_0} &= -2\int_{-\pi}^{\pi}\left(f(x) - a_0 - \sum_{n=1}^{N} a_n \cos(nx)\right)dx \\
&= -2\int_{-\pi}^{\pi} f(x)dx + 2a_0 \underbrace{\int_{-\pi}^{\pi} dx}_{=2\pi} + 2\int_{-\pi}^{\pi}\sum_{n=1}^{N} a_n \cos(nx)dx \\
&= -2\int_{-\pi}^{\pi} f(x)dx + 4\pi a_0 + 2\sum_{n=1}^{N}\underbrace{\int_{-\pi}^{\pi} a_n \cos(nx)dx}_{=0} \\
&= -2\int_{-\pi}^{\pi} f(x)dx + 4\pi a_0.
\end{aligned}$$

Setting $\frac{\partial E_N}{\partial a_0}$ to 0 gives

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx.$$

Grading manual (5p) for correctly identifying a_k $(k = 1, 2, 3, \ldots)$ from $\frac{\partial E_N}{\partial a_k} = 0$ and (5p) for correctly identifying a_0 from $\frac{\partial E_N}{\partial a_0} = 0$.

Problem 3 Fourier series [10 pts] (4D)

Find the Fourier coefficients of the 2-periodic function defined, for $x \in [-1, 1]$, as $f(x) = |x|+1$. Explicitly write down the first 3 non-vanishing terms of the series.

Solution Since $f(x)$ is even, we know that $b_n = 0$, and that

$$a_0 = \frac{1}{L}\int_0^L f(x)\,\mathrm{d}x = \frac{1}{1}\int_0^1 1 + x\,\mathrm{d}x = \frac{3}{2}$$

and

$$\begin{aligned} a_n &= \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,\mathrm{d}x \\ &= 2\int_0^1 (1+x)\cos n\pi x\,\mathrm{d}x \\ &= 2\int_0^1 x\cos n\pi x\,\mathrm{d}x \\ &= 2\left\{x\,\frac{\sin n\pi x}{n\pi}\bigg|_0^1 - \int_0^1 1\cdot\frac{\sin n\pi x}{n\pi}\,\mathrm{d}x\right\} \\ &= \frac{2}{(n\pi)^2}\cos n\pi x\bigg|_0^1 \\ &= 2\frac{(-1)^n - 1}{(n\pi)^2} \end{aligned}$$

We therefore have $a_0 = 1.5$, $a_1 = -4/\pi^2$, $a_2 = 0$, $a_3 = -4/(3\pi)^2$... hence:

$$f(x) = \frac{3}{2} - \frac{2}{\pi^2}\sum_{n=1}^{\infty}\frac{1+(-1)^{n+1}}{n^2}\cos n\pi x = \frac{3}{2} - \frac{4}{\pi^2}\cos n\pi x - \frac{4}{9\pi^2}\cos 3\pi x + \ldots.$$

Grading manual (10p)

Problem 4 Numerical methods for ODEs [10 pts] The following Python function implements an embedded pair of Runge–Kutta methods

```python
import numpy as np
def BogSham(t0, y0, h, f):
    K1 = f(t0, y0)
    K2 = f(t0 + h/2, y0 + h/2 * K1)
    K3 = f(t0 + 3/4 * h, y0 + 3/4 * h * K2)
    y_1 = y0 + h/9 * (2 * K1 + 3 * K2 + 4 * K3)
    K4 = f(t0 + h, y_1)
    y_1_hat = y0 + h/24 * (7 * K1 + 6 * K2 + 8 * K3 + 3 * K4)
    est = y_1_hat - y_1
    return(y_1, est)
```

a) Fill in the coefficients of this embedded pair in a table (Butcher tableau) of the form

$$
\begin{array}{c|cccc}
0 & 0 & & & \\
c_2 & a_{2,1} & 0 & & \\
\vdots & \vdots & \ddots & \ddots & \\
c_s & a_{s,1} & \cdots & a_{s,s-1} & 0 \\
\hline
 & b_1 & b_2 & \dots & b_s \\
\hline
 & \hat{b}_1 & \hat{b}_2 & \dots & \hat{b}_s
\end{array}
$$

Solution We can simply read off the coefficients from the Python code above

$$
\begin{array}{c|cccc}
0 & 0 & & & \\
\frac{1}{2} & \frac{1}{2} & 0 & & \\
\frac{3}{4} & 0 & \frac{3}{4} & 0 & \\
\hline
 & \frac{2}{9} & \frac{3}{9} & \frac{4}{9} & 0 \\
\hline
 & \frac{7}{24} & \frac{6}{24} & \frac{8}{24} & \frac{3}{24}
\end{array}
$$

b) We now try out the method on the problem

$$y'(t) = \frac{1}{1+\tan^2 y}, \quad y(0) = 0.$$

By running the code

```python
def tanprob(t,y):
    return 1/(1+np.tan(y)**2)

```

```python
t0, y0 = 0.0, 0.0
h=0.5
y_1, est = BogSham(t0, y0, h, tanprob)
print(y_1,est)
```

we get the output

`y_1= 0.46337618091961313 , est= 0.0033304765892552 51`

The variable `est` returned by `BogSham` is an error estimate that behaves approximately as $\texttt{est} \approx Ch^3$ for some unknown constant C.

Use this information to estimate a step size h_1 which will make the value of `est` in the succeeding step approximately equal to $\texttt{tol} = 10^{-4}$.

Solution We have $\texttt{est} \approx Ch^3$ and $\texttt{est}_1 \approx Ch_1^3$ so we eliminate C and solve for h_1 to get, with $\texttt{est}_1 = \texttt{tol} = 10^{-4}$

$$h_1 \approx \left(\frac{\texttt{est}_1}{\texttt{est}}\right)^{1/3} h \approx \left(\frac{10^{-4}}{0.0033305}\right)^{1/3} \cdot 0.5 \approx 0.1554$$

Grading manual

a) (5p)

b) (5p)

Problem 5 Interpolation [10 pts]

We consider the polynomial $p(x)$ of lowest degree which interpolates the values

x_i	-2	-1	1	2
y_i	0	1	-1	0

.

a) Explain why the resulting polynomial is odd without computing it.

b) Compute the interpolating polynomial.

Solution. The interpolating polynomial p is of degree not larger than three. To show that p is odd, we consider the polynomial $q(x) = -p(-x)$. It interpolates the same values as p, $q(-2) = -p(2) = 0$, $q(-1) = -p(1) = 1$, $q(1) = -p(-1) = -1$, and $q(-2) = -p(2) = 0$. On the other hand we know there there is only one polynomial of degree not larger than three that is interpolating for the values at four distinct points. Both $p(x)$ and $q(x)$ are such interpolating polynomials. Thus $p(x) = q(x) = -p(-x)$ and p is odd.

We use the Lagrange interpolating formula to compute $p(x)$. The cardinal functions are

$$\begin{aligned} l_1 &= \frac{(x+1)(x-1)(x-2)}{(-2+1)(-2-1)(-2-2)} = -\frac{1}{7}(x+1)(x-1)(x-2), \\ l_2 &= \frac{(x+2)(x-1)(x-2)}{(-1+2)(-1-1)(-1-2)} = \frac{1}{6}(x+2)(x-1)(x-2), \\ l_3 &= \frac{(x+2)(x+1)(x-2)}{(1+2)(1+1)(1-2)} = -\frac{1}{6}(x+2)(x+1)(x-2), \\ l_4 &= \frac{(x+2)(x+1)(x-1)}{(2+2)(2+1)(2-1)} = \frac{1}{7}(x+2)(x+1)(x-1). \end{aligned}$$

Then

$$\begin{aligned} p(x) = l_2 - l_3 &= \frac{1}{6}\left(((x+2)(x-1)(x-2) - (x+2)(x+1)(x-2)\right) \\ &= \frac{1}{6}(x+2)(x-2)(x-1+x+1) = \frac{1}{3}(x+2)(x-2)x = \frac{x^3}{3} - \frac{4x}{3}. \end{aligned}$$

Grading manual

a) (5p)

b) (5p)

Problem 6 Heat Equation [10 pts]

The steady state temperature of the two-dimensional plate is modeled by the equation $u_{xx} + u_{yy} = 0$. We consider this equation on the square $[0,1] \times [0,1]$.

a) Find all solutions of the form $u(x,y) = F(x)G(y)$ that satisfy the boundary conditions

$$u_x(0,y) = u_x(1,y) = 0.$$

b) Find all solutions of the boundary value problem

$$u_{xx} + u_{yy} = 0, \quad u_x(0,y) = u_x(1,y) = 0,\ u_y(x,0) = 0,\ u_y(x,1) = \cos 5\pi x$$

Solution a) For solutions of the form $u(x,y) = F(x)G(y)$ we get the following equation $F''(x)G(y) + F(x)G''(y) = 0$. Separating the variables, we obtain $F''(x) = -kF(x)$ and $G''(y) = kG(y)$. To satisfy the boundary conditions $u_x(0,y) = u_x(1,y) = 0$ we should have $F'(0) = F'(1) = 0$.

When $k = 0$ we get a constant solution $F_0(x) = C_0$ and $G_0(y) = ay+b$, and $u_0(x,y) = A_0y+B_0$.

When $k = w^2 > 0$ we obtain $F(x) = C_1 \cos wx + C_2 \sin wx$ and $F'(x) = -wC_1 \sin wx + C_2 w \cos wx$. The condition $F'(0) = F'(1) = 0$ implies that $C_2 = 0$ and if $F \neq 0$ then $w = n\pi$ for some $n = 1, 2, \ldots$. We have $w_n = n\pi$, $F_n = C_n \cos n\pi x$ and $G_n(y) = A_n e^{w_n y} + B_n e^{-w_n y}$. Then

$$u_n(x,y) = \cos n\pi x(A_n e^{w_n y} + B_n e^{-w_n y}),\ w_n = n\pi, n = 1, 2, \ldots.$$

For $k = -\kappa^2 < 0$ we have $F(x) = C_1 e^{\kappa x} + C_2 e^{-\kappa x}$, $F'(x) = C_1 \kappa e^{\kappa x} - C_2 \kappa e^{-\kappa x}$ and the condition $F'(0) = F'(1) = 0$ implies that $F' = 0$. We obtain only the trivial solution $F = 0$.

b) To satisfy the two other boundary conditions we first look for solutions with separated variables which satisfy $G'(0) = 1$. We have $G_0(y) = A_0 y + B_0$ and $G_0'(0) = A_0$, thus we get the solution $G_0 = B_0$ and $u_0(x,y) = C_0$ is a constant solution.

For $n = 1, 2, \ldots$, $G_n(y) = A_n e^{w_n y} + B_n e^{-w_n y}$, $G'(y) = A_n w_n e^{w_n y} - B_n w_n e^{-w_n y}$. Then $G'(0) = 0$ implies that $A_n = B_n$ and we can write $G_n = C_n \cosh w_n y$. The solutions are of the form

$$u_n(x,y) = C_n \cos n\pi x \cosh n\pi y.$$

Using superposition principle, we obtain solutions

$$u(x,y) = C_0 + \sum_n C_n \cos n\pi x \cosh n\pi y.$$

Finally, to satisfy the condition $u_y(x,1) = \cos 5\pi x$ we compute

$$u_y(x,y) = \sum_n C_n n\pi \cos n\pi x \sinh n\pi y$$

and see that $u_y(x,1) = \sin 5\pi x$ when

$$C_5 = \frac{1}{5\pi \sinh 5\pi}, C_n = 0, n = 1, 2, 3, 4, 6, ...,$$

and C_0 is arbitrary. Then

$$u(x,y) = C_0 + \frac{1}{5\pi \sinh 5\pi} \cos 5\pi x \cosh 5\pi y.$$

Grading manual

a) (5p)

b) (5p)

Problem 7 Wave Equation [10 pts]

The d'Alembert formula for solutions of the wave equation $u_{tt} = c^2 u_{xx}$ can be written in the form

$$u(t,x) = \frac{1}{2}(f(x+ct) + f(x-ct)) + (g * h_t)(x),$$

where $h_t = 1/(2c)$ on $[-ct, ct]$ and $w_t = 0$ otherwise.

a) Express the Fourier transform $\hat{u}(t,w)$ of $u(t,x)$ with respect to the variable x in terms of the Fourier transforms of f and g.

b) Show that $v(t,w) = \hat{u}(t,x)$ satisfies the equation $v_{tt} = -c^2 w^2 v$ with the initial conditions $v(0,w) = \hat{f}(w)$ and $v_t(0,w) = \hat{g}(w)$.

Solution a) First we compute the Fourier transform of h_t. We have

$$\hat{h}_t(w) = \frac{1}{\sqrt{2\pi}} \int_{-ct}^{ct} \frac{1}{2c} e^{-iwx} dx = \frac{1}{-2iwc\sqrt{2\pi}} \left(e^{-icwt} - e^{icwt}\right) = \frac{\sin xwt}{\sqrt{2\pi}wc}.$$

We will use the shift rule for the Fourier transform, $\mathcal{F}(f(x-a))(w) = e^{-iaw}\hat{f}(w)$ and the convolution rule for the Fourier transform $\mathcal{F}(f * h) = \sqrt{2\pi}\mathcal{F}(f)\mathcal{F}(h)$. Then we get

$$\hat{u}(t,w) = \frac{1}{2}\hat{f}(w)(e^{-ictw} + e^{ictw}) + \sqrt{2\pi}\frac{2\sin cwt}{2\sqrt{2\pi}wc}\hat{g}(w) = \hat{f}(w)\cos ctw + \hat{g}(w)\frac{\sin cwt}{cw}.$$

b) From the first part of the problem, we see that

$$v(t,w) = \hat{f}(w)\cos ctw + \hat{g}(w)\frac{\sin cwt}{cw}.$$

Then $v(0,w) = \hat{f}(w)$ and $v_t(0,w) = \hat{g}(w)$. Moreover,

$$v_{tt}(t,w) = -c^2 w^2 \hat{f}(w)\cos ctw - cw\hat{g}(w)\sin ctw = -c^2 w^2 v(t,w).$$

Grading manual

a) (5p)

b) (5p)

Problem 8 Discrete Fourier Transform [10 pts]

Let $c = (c_0, c_1, ..., c_{N-1})$ be the discrete Fourier transform of the signal $f = (f_0, f_1, ..., f_{N-1})$. We use the notation $w = e^{2\pi i/N}$. Let $\tilde{f} = (f_1, ..., f_{N-1}, f_0)$ be the cyclic shift of the initial signal f. Prove that the discrete Fourier transform of $\tilde{f}$ is given by $\tilde{c} = (\tilde{c}_1, ..., \tilde{c}_N)$ with $\tilde{c}_j = w^j c_j$.

Solution We use the convention $f_N = f_0$. By the definition of the discrete Fourier transform,

$$\tilde{c}_j = \frac{1}{N}\sum_{k=0}^{N-1} w^{-jk}\tilde{f}_k = \frac{1}{N}\sum_{k=0}^{N-1} w^{-jk} f_{k+1} = \frac{w^j}{N}\sum_{k=0}^{N-1} w^{-j(k+1)} f_{k+1}.$$

In the last sum we consider the term corresponding to $k = N-1$. We have $w^{-jN} f_N = e^{-2\pi ij} f_0 = f_0$. Thus

$$\tilde{c}_j = w^j \frac{1}{N}\left(\sum_{k=1}^{N-1} w^{-jk} f_k + f_0\right) = w^j c_j.$$

Grading manual (10p)

Problem 9 Numerical integration [10 pts]

The function $f(x) = \sin(\cos(\pi x))$ is 2-periodic, but most of its Fourier coefficients can only be computed numerically. Using the Gauss–Legendre quadrature described below for $\xi \in [-1, 1]$, approximate the coefficient a_1.

i	ξ_i	w_i
1	$-\sqrt{3/5}$	$5/9$
2	0	$8/9$
3	$\sqrt{3/5}$	$5/9$

Remark: The Gauß Legendre quadrature formula of a function $g : [-1, 1] \to \mathbb{R}$ is given by $Q[g] = \sum_{i=1}^{3} w_i g(\xi_i)$.

Solution

Since the integration interval is $[-1, 1]$, we can use the data from the table as it is:

$$\begin{aligned}
a_1 &= \frac{1}{1}\int_{-1}^{1} [\sin(\cos\pi x)]\cos\pi x \,\mathrm{d}x \\
&\approx \frac{5}{9}\sin(\cos(-\pi\sqrt{0.6}))\cos(-\pi\sqrt{0.6}) + \frac{8}{9}\sin(\cos 0)\cos 0 + \frac{5}{9}\sin(\cos(\pi\sqrt{0.6}))\cos(\pi\sqrt{0.6}) \\
&\approx 2 \times \frac{5}{9}(-0.76)\sin(-0.76) + \frac{8}{9}\sin(1) \\
&\approx 0.8642
\end{aligned}$$

Alternative solution

Since $f(x)$ is an even function, we could also write

$$a_1 = 2\int_{0}^{1} [\sin(\cos\pi x)]\cos\pi x \,\mathrm{d}x$$

In this case, we have to transform each ξ_i into a corresponding x_i using

$$x(\xi) = \frac{1-0}{2}\xi + \frac{1+0}{2} = \frac{1+\xi}{2}$$

This gives us $x_1 \approx 0.1127$, $x_2 = 0.5$, $x_3 \approx 0.8873$ and $\mathrm{d}\xi = 2\mathrm{d}x$. Hence:

$$\begin{aligned}
a_1 &= 2\int_0^1 [\sin(\cos \pi x)] \cos \pi x \, \mathrm{d}x \\
&= \int_{-1}^1 [\sin(\cos \pi x)] \cos \pi x \, \mathrm{d}\xi \\
&\approx \frac{5}{9} \sin(\cos(0.1127\pi)) \cos(0.1127\pi) + \frac{8}{9} \sin\left(\cos \frac{\pi}{2}\right) \cos \frac{\pi}{2} + \frac{5}{9} \sin(\cos(0.8873\pi)) \cos(0.8873\pi) \\
&\approx 0.8404
\end{aligned}$$

Grading manual (10p)

Problem 10 Numerics for Nonlinear Equations [10 pts]

Consider the nonlinear equation

$$x\sin(\pi x) - \cos(\pi x) = 0\,.$$

a) Show that this equation has at least one root $r \in (0,1)$

b) Knowing that r is the only root of the equation in the interval $(0,1)$, how many iterations of the bisection method will be needed, at most, to guarantee an absolute error not larger than 2^{-10}?

Solution

a) Solving the nonlinear equation is the same as finding the roots of the function $f(x) = x\sin(\pi x) - \cos(\pi x)$. Since $f(0)f(1) = -1 < 0$ and $f(x)$ is continuous, we know that there is at least one root $r \in (0,1)$ (because f changes its sign between $x = 0$ and $x = 1$).

b) We can simply use the formula

$$k = \log_2\left(\frac{b-a}{2\text{tol}}\right) = \log_2\left(\frac{1-0}{2\times 2^{-10}}\right) = 9 \text{ iterations.}$$

Grading manual

a) (8p)

b) (2p)

Grading Scale (following https://i.ntnu.no/wiki/-/wiki/English/Grading+scale+using+percentage+points):

A	80–90
B	69–79
C	59–68
D	48–58
E	37–47
F	0–36