Problem 1.

Using the Laplace transform, solve the ordinary differential equation

$$2y'' + y' - y = 3u(t-2)\,,$$

with initial conditions $y(0) = y'(0) = 0$, and with u denoting the Heaviside function.

Problem 2.

Using the Laplace transform, solve the integro-differential equation

$$y'(t) + 5\int_0^t \mathrm{e}^{2\tau} y(t-\tau)\,\mathrm{d}\tau = \mathrm{e}^{2t}\,,$$

with the initial condition $y(0) = 0$.

Problem 3.

The function $f\colon [0,1] \longrightarrow \mathbb{R}$ given by

$$f(x) = 2 + x \qquad \text{for } 0 \le x \le 1$$

is to be extended to an odd function g with period 2. Sketch the graph of g on the interval $[-3,3]$ and compute the Fourier series of g.

Problem 4.

Find all non-trivial solutions of the equation

$$\frac{\partial u}{\partial t} = 4\frac{\partial^2 u}{\partial x^2} \qquad \text{where } -1 < x < 1 \quad \text{and } t > 0$$

that are of the form $u(x,t) = F(x)G(t)$ and that satisfy the boundary conditions

$$u(-1,t) = 0 \qquad \text{and} \qquad u(1,t) = 0 \qquad \text{for } t > 0.$$

Problem 5.

Use the Fourier transform in order to solve the PDE

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x} + u \qquad \text{for } x \in \mathbb{R} \text{ and } t > 0$$

with initial conditions

$$u(x,0) = \begin{cases} \dfrac{\sin(x)}{x} & \text{if } x \neq 0, \\ 1 & \text{if } x = 0. \end{cases}$$

Problem 6. (4N only)

Consider the three-point finite-difference formula below:

$$\frac{-3u(x) + 4u(x+h) - u(x+2h)}{2h} = u'(x) + O(h^2)\,.$$

Using a computer with machine accuracy $\varepsilon > 0$, the approximation of u' will have, in addition to the **truncation error** $O(h^2)$, a **rounding error** dependent on ε and h. These two errors will have comparable orders of magnitude when

$$h = O(\varepsilon^p)\,, \quad p > 0.$$

Determine the value of p.

Problem 6. (4D only)

Classify the PDE

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u\,\frac{\partial u}{\partial x}$$

as linear or nonlinear.

Show that, for given constants a and b,

$$u(x,t) = a + \frac{2}{at + x + b}$$

is a solution of the PDE.

Problem 7.

Consider the fixed-point equation $x = \sqrt{x} + 2$, which has a unique real solution r.

a) Show that the fixed-point iterations $x_{k+1} = \sqrt{x_k} + 2$ will converge to r, for any initial guess $x_0 \in [1, 9]$.

b) Starting from $x_0 = 2$, determine how many iterations, at most, will be needed until having

$$|x_{k+1} - r| \leq 2^{-10} .$$

Important: you are *not* being asked to perform these iterations!

Problem 8.

A Runge–Kutta method for solving a scalar ODE, $y' = f(t, y)$, is implemented, the main block of the code is given below:

```
for n in range(N):
    yn = y[n]
    tn = t[n]
    k1 = f(tn, yn)
    k2 = f(tn+0.5*h,yn+0.5*h*k1)
    k3 = f(tn+h, yn+0.5*h*(k1+k2))
    y[n+1] = yn+h/6*(k1+4*k2+k3)
```

Write down the Butcher-tableau, and determine the order of the method.

Problem 9.

We are given the following embedded Runge-Kutta pair:

$$\begin{array}{c|ccc}
0 & 0 & 0 & 0 \\
1/2 & 1/2 & 0 & 0 \\
3/4 & 0 & 3/4 & 0 \\
\hline
 & 0 & 1 & 0 \\
 & 2/9 & 1/3 & 4/9
\end{array} .$$

The first row of b coefficients gives a second order accurate method, and the second row gives a method of order three. The pair is applied to the ODE

$$y' = -y^2, \quad y(0) = 1.$$

a) Let the initial step size be $h_0 = 0.2$ and perform one step with the highest order method.

b) Let y_1, $\hat{y}_1$ denote the solutions after one step with stepsize $h_0 = 0.2$ with the lowest, respectively the highest order method, both starting from $y(0) = y_0 = 1$.

Compute the local error estimate $\hat{\epsilon}_1 = |y_1 - \hat{y}_1|$.

c) Comparing $\hat{\epsilon}_1$ with the tolerance tol $= 10^{-3}$, check if the first step is acceptable, and if not, compute a new stepsize h_{new}. Use the pessimist factor (safety factor) $P = 0.8$.

Problem 10.

Given the two point boundary value problem

$$u'' + x^2 u = \sin(\pi x), \qquad 0 \le x \le 1,$$

with boundary conditions

$$u'(0) = 0 \quad \text{and} \quad u(1) = 0.$$

a) Set up a finite difference scheme for this problem using a constant grid size $h = 1/N$.

b) An attempt of an implementation is given below:

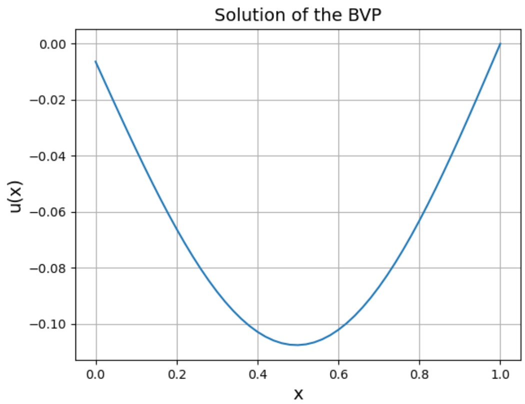
```
import numpy as np

N = 50
h = 1/N
A = np.zeros((N,N))
b = np.zeros(N)
U = np.zeros(N+1)
x = np.linspace(0,1,N+1)

# The coefficient matrix A and the right hand side b
A[0,0] = -2
A[0,1] = 1
for i in range(1,N-1):
    xi = x[i]
    A[i,i-1] = 1
    A[i,i] = -2+h**2*xi**2
    A[i,i+1] = 1
    b[i] = np.sin(np.pi*xi)*h**2
A[N-1,N-2] = 1
A[N-1,N-1] = -2+h**2

U[:-1] = np.linalg.solve(A,b)  # U[N] = 0 is known
```

The plot of the numerical solution is:



How can you conclude that this is *not* the correct solution?

Locate the error in the code.

List of Fourier Transforms.

$f(x)$	$\hat{f}(\omega)$
e^{-ax^2}	$\frac{1}{\sqrt{2a}} e^{-\frac{\omega^2}{4a}}$
$e^{-a\lvert x\rvert}$	$\sqrt{\frac{2}{\pi}} \frac{a}{\omega^2 + a^2}$
$\frac{1}{x^2 + a^2}$ for $a > 0$	$\sqrt{\frac{\pi}{2}} \frac{e^{-a\lvert\omega\rvert}}{a}$
$\begin{cases} 1 & \text{for } \lvert x\rvert < a \\ 0 & \text{otherwise} \end{cases}$	$\sqrt{\frac{2}{\pi}} \frac{\sin(\omega a)}{\omega}$
$\sqrt{\frac{2}{\pi}} \frac{\sin(ax)}{x}$	$\begin{cases} 1 & \text{for } \lvert\omega\rvert < a \\ 0 & \text{otherwise} \end{cases}$

List of Laplace Transforms.

$f(t)$	$F(s)$
$\cos(\omega t)$	$\frac{s}{s^2 + \omega^2}$
$\sin(\omega t)$	$\frac{\omega}{s^2 + \omega^2}$
$\cosh(\omega t)$	$\frac{s}{s^2 - \omega^2}$
$\sinh(\omega t)$	$\frac{\omega}{s^2 - \omega^2}$
t^n	$\frac{n!}{s^{n+1}}$
e^{at}	$\frac{1}{s - a}$
$f(t-a)u(t-a)$	$e^{-sa}F(s)$
$\delta(t-a)$	e^{-sa}

Trigonometric identities.

- $\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$
- $\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$
- $\sin\alpha\cos\beta = \frac{1}{2}\big(\sin(\alpha - \beta) + \sin(\alpha + \beta)\big)$
- $\cos(2\alpha) = 2\cos^2(\alpha) - 1 = 1 - 2\sin^2(\alpha)$
- $2\sin\alpha\cos\beta = \sin(\alpha + \beta) + \sin(\alpha - \beta)$
- $2\cos\alpha\sin\beta = \sin(\alpha + \beta) - \sin(\alpha - \beta)$
- $2\cos\alpha\cos\beta = \cos(\alpha - \beta) + \cos(\alpha + \beta)$
- $2\sin\alpha\sin\beta = \cos(\alpha - \beta) - \cos(\alpha + \beta)$

Integrals.

- $\int x^n \cos ax \,\mathrm{d}x = \frac{1}{a}x^n \sin ax - \frac{n}{a}\int x^{n-1}\sin ax \,\mathrm{d}x$
- $\int x^n \sin ax \,\mathrm{d}x = -\frac{1}{a}x^n \cos ax + \frac{n}{a}\int x^{n-1}\cos ax \,\mathrm{d}x$

Order conditions for Runge-Kutta methods

p	Conditions	p	Conditions
1	$\sum_{i=1}^{s} b_i = 1$	4	$\sum_{i=1}^{s} b_i c_i^3 = \frac{1}{4}$
2	$\sum_{i=1}^{s} b_i c_i = \frac{1}{2}$		$\sum_{i=1}^{s}\sum_{j=1}^{s} b_i c_i a_{ij} c_j = \frac{1}{8}$
3	$\sum_{i=1}^{s} b_i c_i^2 = \frac{1}{3}$		$\sum_{i=1}^{s}\sum_{j=1}^{s} b_i a_{ij} c_j^2 = \frac{1}{12}$
	$\sum_{i=1}^{s}\sum_{j=1}^{s} b_i a_{ij} c_j = \frac{1}{6}$		$\sum_{i=1}^{s}\sum_{j=1}^{s}\sum_{k=1}^{s} b_i a_{ij} a_{jk} c_k = \frac{1}{24}$