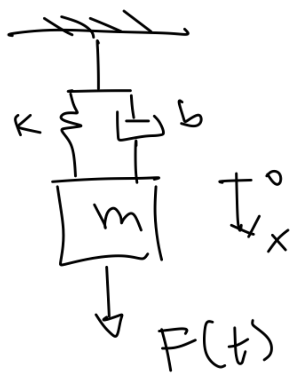
Solving ODEs with the Laplace transform

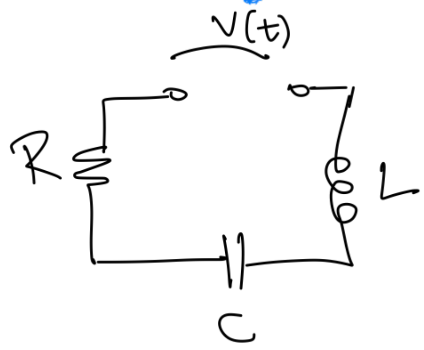
* Remember (existence of the Laplace transform): $f(t)$ has to be (at least) piecewise continuous and there have to exist to non-negative constants a, b such that $\boxed{|f(t)| \leq a e^{bt}}$ for all $t \geq 0$

→ Counter-example: $f(t) = e^{t^2}$ does not have a Laplace transform.

Idea: $\mathcal{L}[f(t)] = \int_0^\infty e^{-st} \cdot f(t)\, dt = \int_0^\infty e^{-st} \cdot e^{t^2} dt = \infty$ (does not exist!)

• Motivation: solving differential and integro-differential equations





$$m x''(t) + b x'(t) + K x(t) = F(t)$$

input force

↳ displacement (output)

$$L i'(t) + R i(t) + \frac{1}{C}\int_0^t i(\tau)\, d\tau = V(t)$$

input voltage

electric current (output)

- Laplace transform of derivatives and integrals

* Recall: $\mathcal{L}[f(t)] = \int_0^\infty e^{-st} f(t)\,dt$

↳ what about $\mathcal{L}[f'(t)]$ or $\mathcal{L}\left[\int_0^t f(\tau)\,d\tau\right]$?

$$\to \mathcal{L}[f'(t)] = \int_0^\infty e^{-st} f'(t)\,dt = e^{-st} f(t)\Big|_{t=0}^{t\to\infty} - \int_0^\infty -s e^{-st} f(t)\,dt = 0 - \overset{1}{e^{0}} f(0) + s\underbrace{\int_0^\infty e^{-st} f(t)\,dt}_{\mathcal{L}[f(t)]}$$

$$= s\mathcal{L}[f(t)] - f(0) = sF(s) - \boxed{f(0)}$$ → we need to know the initial value of $f(t)$ ✓

$$\to \boxed{\mathcal{L}[f'(t)] = s\mathcal{L}[f(t)] - f(0)}$$ → derivative formula

- But if $\mathcal{L}[f'(t)] = s\mathcal{L}[f(t)] - f(0)$, then

$$\mathcal{L}[f''(t)] = \mathcal{L}[(f'(t))'] = s\mathcal{L}[f'(t)] - f'(0) = s\{s\mathcal{L}[f(t)] - f(0)\} - f'(0)$$

$$= \boxed{s^2\mathcal{L}[f(t)] - sf(0) - f'(0)} = s^2F(s) - sf(0) - f'(0)$$

$$\boxed{\mathcal{L}[f''(t)] = s^2\mathcal{L}[f(t)] - sf(0) - f'(0)}$$ → second derivative formula

- we can use this recursively, to find $\boxed{\mathcal{L}[\underbrace{f^{(n)}}_{\text{n-th derivative}}] = s^n\mathcal{L}[f] - \sum_{j=1}^{n} s^{n-j}\underbrace{f^{(j-1)}(0)}_{\text{(j-1)-th derivative}}}$

→ this means we can write the Laplace transform of any derivative of $f(t)$ (provided that their transforms exist!) in terms of the Laplace transform of $f(t)$ and initial conditions ($f(0), f'(0), f''(0), \dots$).

* <u>Integral</u>: $\mathcal{L}\left[\int_0^t f(\tau)d\tau\right] = ?$

- We have seen that $\mathcal{L}[g'(t)] = s\mathcal{L}[g(t)] - g(0)$. Now, make $g(t) := \int_0^t f(\tau)d\tau$.

→ In that case: $\begin{cases} g'(t) = \dfrac{d}{dt}\left[\int_0^t f(\tau)d\tau\right] = f(t) \\ g(0) = \int_0^0 f(\tau)d\tau = 0 \end{cases}$

→ $\mathcal{L}[\overbrace{g'(t)}^{f(t)}] = s\mathcal{L}[\underbrace{g(t)}_{\int_0^t f(\tau)d\tau}]$, or $\mathcal{L}[f(t)] = s\mathcal{L}\left[\int_0^t f(\tau)d\tau\right]$, that is:

$$\boxed{\mathcal{L}\left[\int_0^t f(\tau)d\tau\right] = \frac{1}{s}\mathcal{L}[f(t)] = \frac{F(s)}{s}}$$

→ Integral formula

Example: Knowing that $\mathcal{L}[\cos\omega t] = \frac{s}{s^2+\omega^2}$, find $\mathcal{L}[\sin\omega t]$

$$\rightarrow (\cos\omega t)' = -\omega\sin\omega t \Rightarrow \boxed{\sin\omega t = -\frac{1}{\omega}(\cos\omega t)'} \Rightarrow \mathcal{L}[\sin\omega t] = \mathcal{L}\left[-\frac{1}{\omega}(\cos\omega t)'\right]$$

$$\Rightarrow \mathcal{L}[\sin\omega t] = -\frac{1}{\omega}\mathcal{L}[(\cos\omega t)'] = -\frac{1}{\omega}\left\{s\mathcal{L}[\underbrace{\cos\omega t}_{f(t)}] - \underbrace{\cos 0}_{f(0)}\right\} = -\frac{1}{\omega}\left\{s\cdot\frac{s}{s^2+\omega^2} - 1\right\}$$

$$= -\frac{1}{\omega}\left\{\frac{s^2-s^2-\omega^2}{s^2+\omega^2}\right\} = \boxed{\frac{\omega}{s^2+\omega^2}}$$

* Exercise: do the same, but now using the integral formula! Determine $\mathcal{L}(\sin\omega t)$

• Solving second-order linear ODEs

$$ay''(t) + by'(t) + cy(t) = f(t), \quad y(0) = y_0 \text{ and } y'(0) = V_0$$

$$\mathcal{L}[ay''(t) + by'(t) + cy(t)] = \mathcal{L}[f(t)]$$

$$a\mathcal{L}[y''(t)] + b\mathcal{L}[y'(t)] + c\mathcal{L}[y(t)] = F(s)$$

$$a[s^2Y(s) - s\overset{y_0}{y(0)} - \overset{V_0}{y'(0)}] + b[sY(s) - \overset{y_0}{y(0)}] + cY(s) = F(s)$$

$$[as^2+bs+c]\,Y(s) = F(s) + a y_0 s + b y_0 + a v_0 \Rightarrow \boxed{Y(s) = \frac{F(s) + a y_0 s + b y_0 + a v_0}{as^2+bs+c}}$$

$$\rightarrow \boxed{y(t) = \mathcal{L}^{-1}[Y(s)]}$$

but how about $y(t)$ now?

• Examples (solving initial value problems)

a) $y'' + y = 0, \; y(0) = 1, \; y'(0) = 0$

second-derivative formula

$$\mathcal{L}[y''+y] = \mathcal{L}(0) = 0 \Rightarrow \mathcal{L}[y''] + \mathcal{L}[y] = 0 \Rightarrow \underbrace{\left(s^2 Y(s) - s\,\underset{1}{y(0)} - \underset{0}{y'(0)}\right)} + Y(s) = 0$$

$$\Rightarrow (s^2+1)\,Y(s) = s \Rightarrow \boxed{Y(s) = \frac{s}{s^2+1}} \rightarrow y(t) = \mathcal{L}^{-1}(Y(s))$$

if we look up on a table of transforms, we will see that

$$\mathcal{L}^{-1}\left[\frac{s}{s^2+1^2}\right] = \cos t \Rightarrow \boxed{y(t) = \cos t}$$

b) $y'' - 6y' + 13y = 0, \quad y(0) = 1, \quad y'(0) = 5$

$\to \mathcal{L}(y'') - 6\mathcal{L}(y') + 13\mathcal{L}(y) = \mathcal{L}(0) = 0$

$$[s^2 Y(s) - s\underbrace{y(0)}_{1} - \underbrace{y'(0)}_{5}] - 6[sY(s) - \underbrace{y(0)}_{1}] + 13Y(s) = 0$$

$$(s^2 - 6s + 13)Y(s) = s - 1 \Rightarrow \boxed{Y(s) = \frac{s-1}{s^2 - 6s + 13}}$$

$\to$ we cannot factorise this denominator using real numbers ($s^2 - 6s + 13$ has no real roots)

$\to$ shifted transforms

$$\to Y(s) = \frac{s-1}{s^2 - 6s + 9 + 4} = \frac{s-1}{(s-3)^2 + 4} = \frac{s-3+2}{(s-3)^2+4} = \frac{s-3}{(s-3)^2+4} + \frac{2}{(s-3)^2+4}$$

$$\to y(t) = \mathcal{L}^{-1}\left[\frac{s-3}{(s-3)^2+2^2}\right] + \mathcal{L}^{-1}\left[\frac{2}{(s-3)^2+2^2}\right] = \mathcal{L}^{-1}[F_1(s-3)] + \mathcal{L}^{-1}[F_2(s-3)],$$

where $\boxed{F_1(s) = \frac{s}{s^2+2^2}}$ and $\boxed{F_2(s) = \frac{2}{s^2+2^2}}$

$\to f_1(t) = \cos 2t$ $\qquad$ $\to f_2(t) = \sin 2t$

$\Rightarrow \mathcal{L}^{-1}[F_1(s-3)] = e^{3t} f_1(t) = e^{3t}\cos 2t$, and $\mathcal{L}^{-1}[F_2(s-3)] = e^{3t} f_2(t) = e^{3t}\sin 2t$

$\to$ Solution: $y(t) = e^{3t}\cos 2t + e^{3t}\sin 2t = e^{3t}(\cos 2t + \sin 2t)$ //

$f(t) \Rightarrow F(s) = 2/s^2$

c) $y''' + y' = 2t$, $y(0) = y'(0) = y''(0) = 0$

$$\mathcal{L}(y''') + \mathcal{L}(y') = \mathcal{L}(2t) = 2\mathcal{L}(t) = 2\cdot\frac{1!}{s^{1+1}} = \frac{2}{s^2}$$

- Because all initial conditions are zero, we will have $\begin{cases} \mathcal{L}(y') = sY(s) \\ \mathcal{L}(y''') = s^3 Y(s) \end{cases}$

$$s^3 Y(s) + sY(s) = \frac{2}{s^2} \Rightarrow \boxed{Y(s) = \frac{2}{s^2(s^3+s)} = \frac{2}{s^3(s^2+1)}}$$

$\to$ Let's factorise the denominator first:

$$Y(s) = \frac{2}{s^3(s-i)(s+i)} \to \text{Partial fractions: } Y(s) = \frac{A}{s^3} + \frac{B}{s^2} + \frac{C}{s} + \frac{D}{s-i} + \frac{E}{s+i}$$

$$\frac{2}{s^3(s-i)(s+i)} = \frac{A}{s^3} + \frac{B}{s^2} + \frac{C}{s} + \frac{D}{s-i} + \frac{E}{s+i} \quad \text{for all } s$$

$$\frac{2}{\cancel{s^3}\cancel{(s-i)}\cancel{(s+i)}} = \frac{A(s^2+1) + Bs(s^2+1) + Cs^2(s^2+1) + Ds^3(s+i) + Es^3(s-i)}{\cancel{s^3}\cancel{(s-i)}\cancel{(s+i)}}$$

$$\boxed{(C+D+E)s^4 + (B+iD-iE)s^3 + (A+C)s^2 + Bs + (A-2) = 0} \quad \text{for all } s$$

$$\Rightarrow \begin{cases} A-2=0 \\ B=0 \\ A+C=0 \\ B+i(D-E)=0 \\ C+D+E=0 \end{cases} \Rightarrow \begin{cases} A=2 \\ B=0 \\ C=-2 \\ D-E=0 \\ D+E=2 \end{cases} \Rightarrow \begin{cases} A=2 \\ B=0 \\ C=-2 \\ D=1 \\ E=1 \end{cases}$$

$$\Rightarrow Y(s) = \frac{2}{s^3} - \frac{2}{s} + \frac{1}{s-i} + \frac{1}{s+i}$$

t^2 -2 ?

* One approach: $\frac{1}{s-i}+\frac{1}{s+i}=\frac{s+i+s-i}{(s-i)(s+i)}=2\cdot\frac{s}{s^2+1} \rightarrow \cos t$

* Alternative: we have a formula for $\frac{1}{s-a}$: $\mathcal{L}^{-1}\left(\frac{1}{s-i}+\frac{1}{s+i}\right)=\underbrace{e^{it}}_{\cos t+i\sin t}+\underbrace{e^{-it}}_{\cos t-i\sin t}=2\cos t$

$\rightarrow$ Final answer: $y(t)=t^2-2+2\cos t$