NTNU

Norwegian University of Science and Technology

TMA4130 Matematikk 4N

Week 40, first lecture:
Partial differential equations

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Partial differential equations

Definition: PDEs are equations written in terms of a function (say, u) and its (partial) derivatives with respect to at least two independent variables.

↳ if we only have 1 independent variable, then that's actually an ODE

↳ ordinary

→ The solution of a differential equation is a function (or multiple function)

→ We will denote our unknown function as u

Important examples (PDEs)

* Heat equation (1D): $\frac{\partial u}{\partial t} - \alpha \frac{\partial^2 u}{\partial x^2} = q(x,t)$

u: temperature
time
space (1D)
heat source

* Wave equation (1D): $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$

c: wave speed

* Vibration of an elastic beam (bar): $\frac{\partial^2 u}{\partial t^2} + k^2 \frac{\partial^4 u}{\partial x^4} = q(x,t)$

elastic parameter
displacement (transverse)
mechanical load

* Deflection of plates: $\frac{\partial^4 u}{\partial x^4} + 2\frac{\partial^4 u}{\partial x^2 \partial y^2} + \frac{\partial^4 u}{\partial y^2} = q(x,y)$ (bi-harmonic equation)

vertical displacement

(static)

The Poisson equation

$\nabla^2 u = f$; in 2D : $\nabla^2 u := \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$; in 3D : $\nabla^2 u := \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$

↳ Laplace operator (Laplacian)

* When the right-hand side f is identically zero ($f \equiv 0$), we call this the Laplace equation ($\nabla^2 u = 0$)

Examples (applications) :

* stationary temperature distribution in solids or fluids
* electric potential
* inviscid flow (negligible viscosity)
* torsion of prismatic bars

...

Linear and nonlinear PDEs

A linear PDE can be written as a linear combination of 1, u, and the partial derivatives of u.

→ The coefficients of this linear combination need not be constant, that is, they can depend on the independent variables, but not on the unknown u.

Ex.: $\frac{\partial u}{\partial t} - 2\frac{\partial u}{\partial x} = 0$; $x^2 \frac{\partial u}{\partial x} - 3\frac{\partial^2 u}{\partial y^2} - 2y^3 \cdot 1 + 10 \cdot u = 0$ (both linear)

Counter-examples: * $\frac{\partial u}{\partial y} - 3\frac{\partial^2 u}{\partial x^2} + \left(\frac{\partial u}{\partial t}\right)^2 = 0$ (non-linear)

* $\frac{\partial^2 u}{\partial x \partial t} + \sin(u) = 0$ (non-linear)

Homogeneous and non-homogeneous PDEs

→ In a homogeneous PDE, all terms depend on u and/or its partial derivatives, and vanish for $u \equiv 0$.

Ex.: $\frac{\partial^2 u}{\partial x \partial t} + \sin(u) = 0$ (homogeneous)

Counter-examples: * $\frac{\partial u}{\partial t} + x\frac{\partial u}{\partial x} = x^2 t$

* $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 1$

→ both non-homogeneous

→ the zero function ($u \equiv 0$) is always (a) solution of a homogeneous PDE

Order of a PDE

→ The order of a PDE is the order of the highest partial derivative appearing in the PDE

Ex.:

* $\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0$ (second-order)

* $x^2 \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = \sin(t)$ (first-order)

* $\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial u}{\partial y} = 0$ (second-order)

* $\frac{\partial u}{\partial x} \cdot \frac{\partial u}{\partial y} = 1$ (first-order)

$\left(\frac{\partial u}{\partial x}\right)^2 + \frac{\partial u}{\partial t} = 1$

(first-order)

Classification of PDEs (linearity, homogeneous/not, order)

Examples:

independent of u

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} - t\frac{\partial^2 u}{\partial x^2} = \left(x\mathrm{e}^{-t^2}\right) \cdot 1$$

(linear, not homogeneous, second-order)

Classification of PDEs

Examples:

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} - t\frac{\partial^2 u}{\partial x^2} = x\mathrm{e}^{-t^2}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0$$ (homogeneous, non-linear, first-order)

Superposition (principle)

Let u_1 and u_2 both be solutions of a certain PDE. If any linear combination $u_3 = \alpha u_1 + \beta u_2$ is also a solution of this PDE, then we say that the PDE admits superposition.

↳ constants

→ the superposition principle holds if, and only if, the PDE is linear and homogeneous.

Initial and boundary conditions

- In general, a PDE can (will) have infinitely many solutions. That's why we usually have boundary and initial conditions.

→ Initial conditions (ICs) are the values of u and its temporal derivatives at the initial time (usually $t=0$)

→ If a PDE is of order k in time (highest time derivative in the PDE is $\frac{\partial^k u}{\partial t^k}$) we will need k ICs: the values of $u, \frac{\partial u}{\partial t}, \dots, \frac{\partial^{k-1} u}{\partial t^{k-1}}$ at $t=0$.

Examples: $\frac{\partial u}{\partial t} - \alpha \frac{\partial^2 u}{\partial x^2} = q(x,t) \rightarrow$ IC : $u(x,0) = f(x)$ (heat)

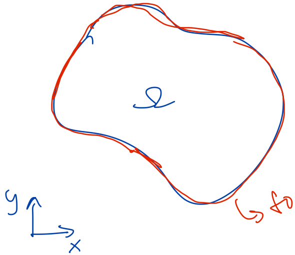
$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0 \rightarrow$ ICs : $u(x,0) = f(x)$ given! (wave)

$\left.\frac{\partial u}{\partial t}\right|_{t=0} = g(x)$

Initial and boundary conditions

– Boundary conditions (BCs) : conditions on u and/or spatial derivatives of u on the boundary of the spatial domain. How many and which ones are needed/possible is a much more complicated matter (depends on the PDE)

Ex.: $\frac{\partial u}{\partial t} - \alpha\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = q(x,y)$

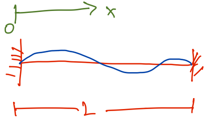


↳ for example : prescribed values of u on the boundary of Ω (domain)

The one-dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$ → energy-conserving wave with speed c

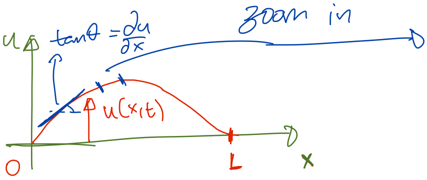
→ Model problem: vibration of a string

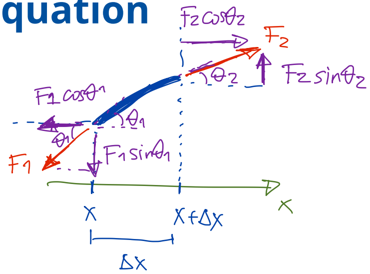


- Modelling assumptions (simplifying assumptions):

* The string is subjected to a constant tension F
* The string has a uniform weight distribution $\left(\frac{\Delta m}{\Delta x} = \frac{m}{L} = \rho \right)$ linear density
* $mg \ll F$ (negligible weight → not mass)
* Lateral motion negligible (only transverse displacements are considered)
* The transverse displacements are small in comparison to L

The one-dimensional wave equation





* No lateral motion: $F_1 \cos\theta_1 = F_2 \cos\theta_2 = F$ (given, constant) (*)

* Newton's second law in the vertical direction (transversal):

$$F_2 \sin\theta_2 - F_1 \sin\theta_1 = \Delta m \frac{\partial^2 u}{\partial t^2} \overset{(*)}{\Longrightarrow} F \overbrace{\frac{\sin\theta_2}{\cos\theta_2}}^{\tan\theta_2} - F \overbrace{\frac{\sin\theta_1}{\cos\theta_1}}^{\tan\theta_1} = \rho \Delta x \frac{\partial^2 u}{\partial t^2} \Longrightarrow$$

$$\Longrightarrow \tan\theta_2 - \underbrace{\tan\theta_1}_{\frac{\partial u}{\partial x}} = \frac{\rho \Delta x}{F} \frac{\partial^2 u}{\partial t^2} \Longrightarrow \boxed{\frac{\rho \Delta x}{F} \frac{\partial^2 u}{\partial t^2} = \left.\frac{\partial u}{\partial x}\right|_{x+\Delta x} - \left.\frac{\partial u}{\partial x}\right|_{x}}$$

The one-dimensional wave equation

- $\frac{\partial u}{\partial x}(x,t) = s(x,t)$

↳ slope of the curve $u(x)$

$$\Rightarrow \boxed{\rho_F \frac{\partial^2 u}{\partial t^2} = \frac{s(x+\Delta x,t) - s(x,t)}{\Delta x}}$$

→ Now, to derive a PDE, we will take the limit for $\Delta x \to 0$

$$\Rightarrow \rho_F \frac{\partial^2 u}{\partial t^2} = \lim_{\Delta x \to 0} \frac{s(x+\Delta x,t) - s(x,t)}{\Delta x} = \frac{\partial s}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial x^2}$$

→ Finally:

$$\boxed{\frac{\partial^2 u}{\partial t^2} = \overbrace{\frac{F}{\rho}}^{c^2} \frac{\partial^2 u}{\partial x^2}}$$

→ wave equation with wave speed c:

$$c = \sqrt{\frac{F}{\rho}}$$