NTNU

Norwegian University of Science and Technology

TMA4130 Matematikk 4N

Week 39, first lecture: The continuous Fourier transform

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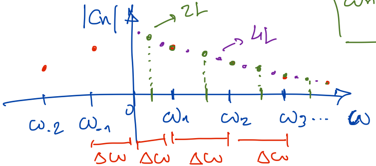
The Fourier spectrum

frequencies

$f(x)$ 2L-periodic $\Rightarrow f(x) = a_0 + \sum_{n=1}^{\infty} a_n\cos(\omega_n x) + b_n\sin(\omega_n x) = \boxed{\sum_{n=-\infty}^{\infty} c_n e^{i\omega_n x}}$

$\boxed{\omega_n = \frac{n\pi}{L}}$



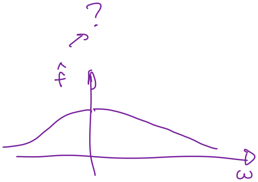
Ex.: $f(x) = x$, 2π-periodic

↳ $c_n = \frac{(-1)^n}{n} i \Rightarrow |c_n| = \frac{1}{|n|}$

– What if $f(x)$ is not periodic at all, and for some reason we want/need to consider $f(x)$ in the whole real axis $x \in (-\infty, \infty)$?

– Let's write $\omega_n = n \cdot \Delta\omega$, $\boxed{\Delta\omega = \frac{\pi}{L}}$

→ As $L \to \infty$, the spectrum will become continuous



The Fourier transform

$$\boxed{f(x)\Big|_{[-L,L]}} = \sum_{n=-\infty}^{\infty} c_n e^{i\omega_n x} = \sum_{n=-\infty}^{\infty} \Big[\overbrace{\frac{1}{2L}\int_{-L}^{L} f(x)\, e^{-i\omega_n x}\, dx}^{c_n}\Big] e^{i\omega_n x} = \sum_{n=-\infty}^{\infty} \Delta\omega \left(\frac{1}{2\pi}\int_{-L}^{L} f(x)\, e^{-i\omega_n x}\, dx\right) e^{i\omega_n x}$$

$$= \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} \Delta\omega \underbrace{\left(\frac{1}{\sqrt{2\pi}}\int_{-L}^{L} f(x)\, e^{-i\omega_n x}\, dx\right)}_{\hat{f}_L(\omega_n)} e^{i\omega_n x} = \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} \underbrace{\hat{f}_L(\omega_n)\, e^{i\omega_n x}}_{g(\omega_n)} \Delta\omega = \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} g(\omega_n)\Delta\omega$$

$$\boxed{\Delta\omega = \frac{\pi}{L}}$$

– Now, we will extend this to $x \in (-\infty, \infty)$, that is,

$$f(x) = \lim_{L\to\infty} f(x)\Big|_{[-L,L]} = \frac{1}{\sqrt{2\pi}} \lim_{L\to\infty} \sum_{n=-\infty}^{\infty} g(\omega_n)\Delta\omega$$

* As $L \to \infty$, $\Delta\omega \to 0$. Therefore: $f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(\omega)\, d\omega = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(\omega)\, e^{i\omega x}\, d\omega$,

where $\hat{f}(\omega) := \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{-i\omega x}\, dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(y)\, e^{-i\omega y}\, dy$ → Continuous Fourier transform of $f(x)$

The Fourier transform

Hence: $f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega x} d\omega = \underbrace{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{\infty} f(y) e^{-i\omega y} dy \right] e^{i\omega x} d\omega}$ $=$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(y) e^{i\omega(x-y)} dy \, d\omega$$

Summarising: $*\ \hat{f}(\omega) = \mathcal{F}(f)(\omega) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx$

$*\ f(x) = \mathcal{F}^{-1}(\hat{f})(x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega x} d\omega$ (inverse transform)

– When does the Fourier transform exist?

↳ If $f(x)$ is absolutely integrable, that is: if $\int_{-\infty}^{\infty} |f(x)| dx < \infty$

The Fourier transform

* Remark: there are, in the literature, different scalings used to define the Fourier transform.

→ Ex.: (alternative definition) $\hat{f}(\omega) = \int_{-\infty}^{\infty} f(x)\, e^{-i\omega x}\, dx$ (without the factor $1/\sqrt{2\pi}$)

Properties of the Fourier transform

* Linearity: $\mathcal{F}(af(x)+bg(x)) = a\mathcal{F}(f) + b\mathcal{F}(g)$ (consequence of the linearity of integration)

* Derivative formula: if $\boxed{\lim_{x\to\pm\infty} f(x) = 0}$ then $\mathcal{F}(f') = i\omega\mathcal{F}(f)$

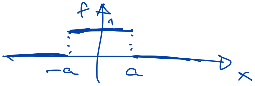
↳ Why?

$$\to \mathcal{F}(f') = \frac{1}{\sqrt{2\pi}} \lim_{L\to\infty} \int_{-L}^{L} f'(x)\, e^{-i\omega x}\, dx \overset{\text{integration by parts}}{=} \frac{1}{\sqrt{2\pi}} \lim_{L\to\infty} \left\{ f(x)\, e^{-i\omega x} \Big|_{-L}^{L} - \int_{-L}^{L} f(x)(-i\omega)\, e^{-i\omega x}\, dx \right\}$$

$$= \frac{1}{\sqrt{2\pi}} \lim_{L\to\infty} \left[\underbrace{f(L)}_{\to 0} e^{-i\omega L} - \underbrace{f(-L)}_{\to 0} e^{i\omega L} \right] + i\omega \underbrace{\lim_{L\to\infty} \frac{1}{\sqrt{2\pi}} \int_{-L}^{L} f(x)\, e^{-i\omega x}}_{\hat{f}(\omega) = \mathcal{F}(f)} = i\omega\mathcal{F}(f)$$

Example

$$f(x) = \begin{cases} 1, & \text{if } |x| < a \\ 0, & \text{otherwise} \end{cases}$$



"indicator function"

– Let's compute the Fourier transform $\hat{f}(\omega)$ of $f(x)$:

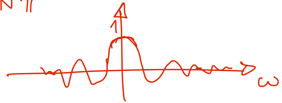
* Note that $f(x)$ is absolutely integrable, since $\int_{-\infty}^{\infty} |f(x)|\,dx = \int_{-a}^{a} |1|\,dx = 2a < \infty$

* $$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x}\,dx = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} 1 \cdot e^{-i\omega x}\,dx = \frac{1}{\sqrt{2\pi}(-i\omega)} e^{-i\omega x} \Big|_{-a}^{a} = \frac{e^{-i\omega a} - e^{+i\omega a}}{\sqrt{2\pi}(-i\omega)} =$$

$$= \frac{[\cos(\cancel{\omega} a) - i\sin(\omega a)] - [\cos(\cancel{\omega} a) + i\sin(\omega a)]}{-\sqrt{2\pi}\, i\omega} = \frac{2\cancel{i}\sin(\omega a)}{\cancel{i}\omega\sqrt{2\pi}} = \sqrt{\frac{2}{\pi}}\, \frac{\sin(\omega a)}{\omega} \quad //$$

→ for example, if $a = 1$ (given), then: $\sqrt{\frac{2}{\pi}}\, \frac{\sin\omega}{\omega} = \sqrt{\frac{2}{\pi}}\, \text{sinc}(\omega)$

↳ $\text{sinc}(\omega)$



Convolution

- Idea: combine two functions to generate a desired output (new function)

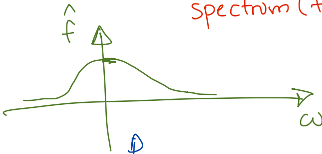
→ Definition: $(f \overset{\text{convolution}}{*} g)(x) = \int_{-\infty}^{\infty} f(y)\, g(x-y)\, dy$ (convolution of f with g)

usual product

- Property: $\mathcal{F}(f*g) = \sqrt{2\pi}\, \mathcal{F}(f) \cdot \mathcal{F}(g) = \sqrt{2\pi}\, \hat{f}(\omega) \cdot \hat{g}(\omega)$

spectrum (the frequencies that compose f)

Ex.:



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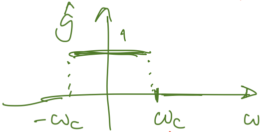
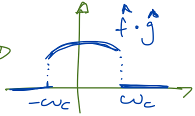


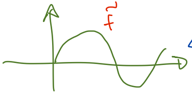
↳ cut-off frequency

$\hat{f} \cdot \hat{g}$ →



$f(x) \xrightarrow{\text{Fourier transform}} \hat{f}(\omega)$

$\tilde{f} = \frac{1}{\sqrt{2\pi}}(f*g)$:

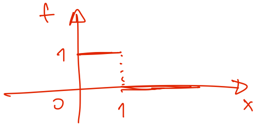


$\mathcal{F}^{-1}(\hat{f} \cdot \hat{g})$ (inverse transform)

$\tilde{f}$: filtered signal

Example

$$f(x) = \begin{cases} 1, & 0 \le x \le 1 \\ 0, & \text{otherwise} \end{cases} \quad ; \quad g(x) = \frac{1}{1+x^2}$$



- Let's compute the convolution $(f*g)(x)$:

* First, notice that both $f(x)$ and $g(x)$ are absolutely integrable (check)

$$\to (f*g)(x) = \int_{-\infty}^{\infty} f(y)\,g(x-y)\,dy = \int_0^1 1 \cdot \frac{1}{1+(x-y)^2}\,dy = \int_0^1 \frac{1}{1+(y-x)^2}\,dy = \underbrace{\operatorname{atan}}_{\tan^{-1}}(y-x)\Big|_{y=0}^{y=1}$$

$$= \operatorname{atan}(1-x) - \operatorname{atan}(-x) = \operatorname{atan}(x) + \operatorname{atan}(1-x) \quad //$$