NTNU

Norwegian University of Science and Technology

TMA4130 Matematikk 4N

Week 35:
Interpolation and numerical integration

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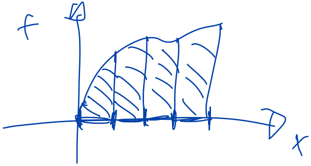
Composite integration rules

Remember our motivation

$$\text{Reactor, } a = 2: \quad u(x) = \frac{u_0}{1 + F(x)}, \quad \text{with} \quad F(x) = \int_0^x \frac{k u_0}{v(x)}\,\mathrm{d}x = {\color{purple}???}$$

$F(x_i) =$



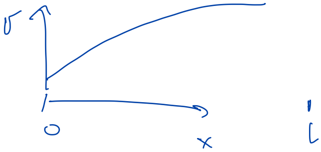
Composite rules are when I certain rule (defined for the interval $[a,b]$) is applied to smaller intervals $[x_i, x_{i+1}]$

Example

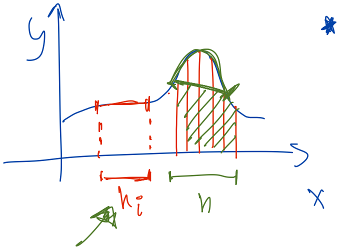
$$u(x) = \frac{u_0}{1 + F(x)}, \quad \text{with} \quad F(x) = \int_0^x \frac{k u_0}{v(x)}\,\mathrm{d}x$$

Problem data: $L = 5$, $u_0 = 1$, $k = 1$, $v(x) = \ln(x + 2)$

$$F(x_i) = \int_0^{x_i} \frac{1}{\ln(x+2)}\,\mathrm{d}x$$



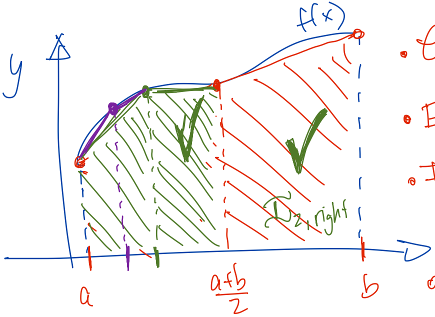
Adaptive integration



* We want to construct an <u>automatic</u> subdivision algorithm for the interval $[a,b]$, based on the integrand. But how to do that?

→ adaptivity

- Compute I_1
- Estimate the integration error (?)
- If the error is too large, we can divide $[a,b]$ into two!
- Then I compute I_2



Adaptive integration

Given f, a, b and a user-defined tolerance Tol. ?

1. Calculate $I(a,b)$ and estimate the error $\mathcal{E}(a,b)$,

2. if $|\mathcal{E}(a,b)| \leq$ Tol

- Accept the result,
- return $I(a,b) + \mathcal{E}(a,b)$ as an approximation to $\int_a^b f(x)\,\mathrm{d}x$.

else

- set $c = (a+b)/2$, and repeat the process on each of the subintervals $[a,c]$ and $[c,b]$, with tolerance Tol$/2$.

3. Sum up the accepted results from each subinterval.

Error estimation

Trapezoidal rule: $I_T - \int_a^b f(x)\,\mathrm{d}x = \frac{{\overbrace{(b-a)}^{L}}^3}{12} f''(\xi)$ if $f \in C^2(a,b)$, $\boxed{\xi \in [a,b]}$

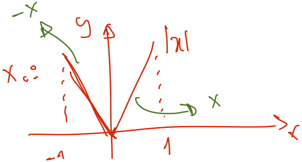
Error estimation

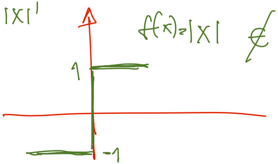
Trapezoidal rule: $I_T - \int_a^b f(x)\,\mathrm{d}x = \frac{(b-a)^3}{12} f''(\xi)$ if $f \in C^2(a,b)$, $\xi \in [a,b]$

Simpson's rule: $I_S - \int_a^b f(x)\,\mathrm{d}x = \frac{(b-a)^5}{2880} \boxed{f^{(4)}(\xi)}$ if $f \in C^4(a,b)$, $\xi \in [a,b]$

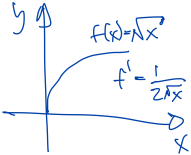
- $f \in C^0(a,b)$ if f is continuous in $[a,b]$ (also denote as $C(a,b)$)
- $f \in C^1(a,b)$ if f and f' are in $C^0(a,b)$ (that is, f and f' are continuous in $[a,b]$)

Ex.:



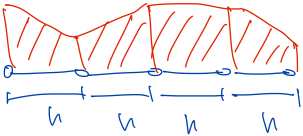


$f(x) = |x| \notin C^1(-1,1)$



- $f \in C^n(a,b)$ if $f, f', \dots, f^{(n)}$ are in $C^0(a,b)$

Error estimation (composite trapz. rule)



$$\mathcal{E} = \frac{(b-a)^3}{12} f''(\varphi)$$

$$\mathcal{E}_i = \frac{h^3}{12} f''(\varphi_i), \quad \varphi_i \in [x_i, x_{i+1}]$$

$$\mathcal{E} = \sum_{i=1}^{n} \mathcal{E}_i = \frac{h^3}{12} \sum_{i=1}^{n} f''(\varphi_i) \quad (*)$$

$\varphi \in [a,b]$

Comp. Simpson

$$\mathcal{E} = -\frac{(b-a) f^{(4)}(\varphi)}{180} h^4$$

$\to \mathcal{O}(h^4)$

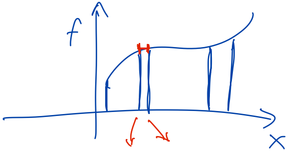
* Generalised mean value theorem: $\sum_{i=1}^{n} f''(\varphi_i) = n f''(\varphi)$, $f''(\varphi) = \frac{\sum_{i=1}^{n} f''(\varphi_i)}{n}$

$$\Rightarrow (*) \quad \mathcal{E} = \frac{h^3}{12} n f''(\varphi) = \frac{h^2}{12} \underbrace{(nh)}_{b-a} f''(\varphi) = \frac{(b-a) f''(\varphi)}{12} h^2 = \boxed{C h^2 \approx \mathcal{E}}$$

Conclusion: the composite trap. rule converges quadratically ($\mathcal{O}(h^2)$)

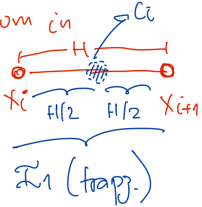
Error estimation (adaptive trapz.)

$$\boxed{\mathcal{E} = \frac{(b-a)^3}{12} f''(\varphi)} \Rightarrow \frac{(x_{i+1}-x_i)^3}{12} f''(\varphi_i)$$



$x_{i+1} - x_i = H$

Zoom in



*Assumption: H is small enough that f'' is approximately constant.

* I_1 : quadrature result in $[x_i, x_{i+1}]$

* I_2 : quadrature results in $[x_i, c_i]$ and $[c_i, x_{i+1}]$ (added)

Error estimates :

$$\begin{cases} I_1 - \int_{x_i}^{x_{i+1}} f\,dx = \dfrac{H^3 f''}{12} \approx CH^3 \\ \underbrace{I_2 - \int_{x_i}^{x_{i+1}} f\,dx}_{\mathcal{E}_2 = ?} = 2\dfrac{(H/2)^3 f''}{12} \approx \dfrac{CH^3}{4} \end{cases}$$

C

(−)

Known

$$\Rightarrow I_1 - I_2 \approx \frac{3}{4} CH^3$$

$$CH^3 \approx \frac{4}{3}(I_1 - I_2)$$

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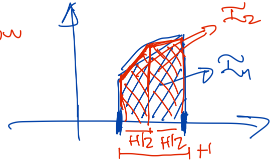
Error estimation

$$\mathcal{E}_2 = \mathcal{I}_2 - \int_{x_i}^{x_{i+1}} f\,dx \approx \frac{1}{4} C H^3 \approx \frac{1}{4}\frac{4}{3}(\mathcal{I}_1 - \mathcal{I}_2) = \boxed{\frac{\mathcal{I}_1 - \mathcal{I}_2}{3} = E_2}$$

actual error → cannot know in practice



We can compute, because we have $\mathcal{I}_2$ and $\mathcal{I}_1$ (results of the quadrature) → error estimator

* E_2 will be compared to a give tolerance "tol" to determine whether our numerical integral (within $[x_i, x_{i+1}]$) is accurate enough

* Mind that $\int_{x_i}^{x_{i+1}} f\,dx = \mathcal{I}_2 - \mathcal{E}_2 \approx \mathcal{I}_2 - E_2$. that is, we get "for free" an even better approximation for $\int_{x_i}^{x_{i+1}} f\,dx$ than simply $\mathcal{I}_2$.