Exercise session

• Spring 2019 (adapted)

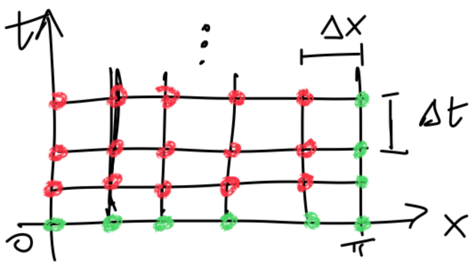
$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}, \quad 0 < x < \pi, \quad t > 0 \\ u(x,0) = \sin^2 x \quad \text{(initial condition)} \\ u(\pi, t) = 0 \quad \text{(Dirichlet BC)} \\ \underbrace{u_x}_{\frac{\partial u}{\partial x}}(0,t) = 0 \quad \text{(Neumann BC)} \end{cases}$$

* Set up the Crank–Nicolson scheme for the numerical solution of this problem.

Solution:

$$U_i^n \approx u(x_i, t_n)$$

not an exponent!



* Define Δt and set $t_n = n\Delta t$, $n = 0, 1, 2, \dots$

* Define N and set $\Delta x = \frac{\pi}{N}$ and $x_i = i\Delta x$, $i = 0, 1, 2, \dots, N$.

N: number of spatial segments

Crank-Nicolson: $\dfrac{\partial u}{\partial t} = \dfrac{1}{2}\dfrac{\partial^2 u}{\partial x^2} + \dfrac{1}{2}\dfrac{\partial^2 u}{\partial x^2}$ → at time t_{n+1}

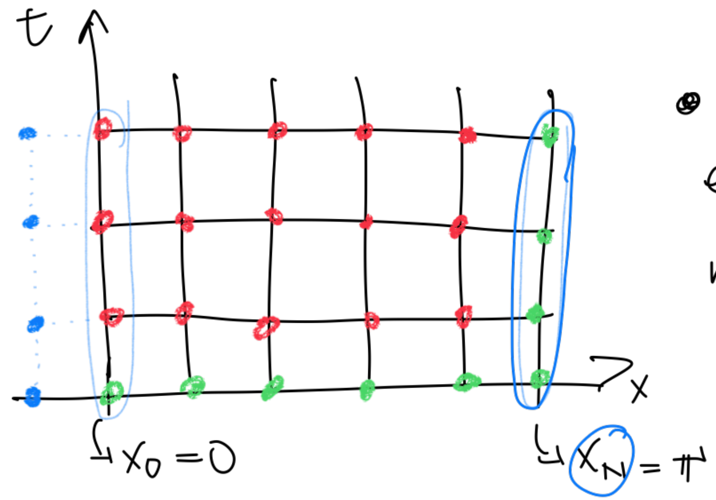
at time t_n

$$\frac{U_i^{n+1} - U_i^n}{\Delta t} = \frac{1}{2}\left(\frac{U_{i-1}^n - 2U_i^n + U_{i+1}^n}{\Delta x^2}\right) + \frac{1}{2}\left(\frac{U_{i-1}^{n+1} - 2U_i^{n+1} + U_{i+1}^{n+1}}{\Delta x^2}\right)$$

– Unknowns: quantities at t_{n+1}:

$$\underbrace{U_i^{n+1}}_{\text{unknown}} - \underbrace{U_i^n}_{\text{known (past)}} = \alpha\underbrace{\left(U_{i-1}^n - 2U_i^n + U_{i+1}^n\right)}_{\text{known (past)}} + \alpha\underbrace{\left(U_{i-1}^{n+1} - 2U_i^{n+1} + U_{i+1}^{n+1}\right)}_{\text{unknown}}, \quad \alpha = \frac{\Delta t}{2\Delta x^2}$$

$$\boxed{-\alpha U_{i-1}^{n+1} + (1+2\alpha)U_i^{n+1} - \alpha U_{i+1}^{n+1} = \alpha U_{i-1}^n + (1-2\alpha)U_i^n + \alpha U_{i+1}^n}, \quad i = 0, 1, \dots, N-1$$



- Setting $i=0$ produces a term U_{-1}^{n+1}, which does not exist a priori (not part of the problem). We thus need an equation for U_{-1}^{n+1}. Neumann BC:

$\dfrac{\partial u}{\partial x} = 0$ for $x = x_0$

→ central discretisation: $\dfrac{U_1^{n+1} - U_{-1}^{n+1}}{2\Delta x} = 0$

$\Rightarrow U_{-1}^{n+1} = U_1^{n+1}$

- The right (hand side) BC is simpler: $\overbrace{U_N^{n+1} = 0}^{\text{Dirichlet}}$ for all $n = 0, 1, \ldots$

- It remains to handle the initialisation $(n=0)$:

$U_i^0 = \sin^2(x_i)$ for $i = 0, 1, \ldots, N$

Final answer:

$$\boxed{\begin{array}{l}
\bullet\ U_i^0 = \sin^2 x_i, \quad \text{for } i = 0, 1, \ldots, N \text{ (all points in space)} \\
\bullet\ U_1^{n+1} - U_{-1}^{n+1} = 0, \quad \text{for } n = 0, 1, \ldots \text{ (all points in time)} \\
\bullet\ -\alpha U_{i-1}^{n+1} + (1+2\alpha) U_i^{n+1} - \alpha U_{i+1}^{n+1} = \alpha U_{i-1}^{n} + (1-2\alpha) U_i^{n} + \alpha U_{i+1}^{n}, \quad \text{for } \begin{cases} i = 0, 1, \ldots, N-1 \\ n = 0, 1, 2, \ldots \end{cases} \\
\bullet\ U_N^{n+1} = 0 \quad \text{for } n = 0, 1, 2, \ldots \text{ (all points in time)}
\end{array}}$$

$U_N^{n+1} \approx u(x_N, t_{n+1}) = u(\pi, t_{n+1}) = 0$

• Autumn 2015 (4D)

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} - 4u \,, \quad 0 < x < \pi \,, \quad t > 0$$

a) Find all non-trivial (non-zero) solutions of the form $\boxed{u(x,t) = F(x)G(t)}$ that fulfill the boundary conditions : $\begin{cases} u = 0 \text{ for } x = 0 \\ \frac{\partial u}{\partial x} = 0 \text{ for } x = \pi \end{cases}$ (for all t)

Solution:

$$\frac{\partial}{\partial x^2}\left(F(x)G(t)\right) = \frac{\partial}{\partial t}\left(F(x)G(t)\right) - 4F(x)G(t) \Rightarrow \frac{\cancel{G}F''}{F\cancel{G}} = \frac{\cancel{F}\dot{G} - 4\cancel{F}G}{\cancel{F}G} \Rightarrow \underbrace{\frac{F''}{F}}_{f(x)} = \underbrace{\frac{\dot{G} - 4G}{G}}_{g(t)} = K$$

ODEs: $\begin{cases} F''(x) = KF(x) \\ \dot{G}(t) - 4G(t) = KG(t) \end{cases} \Rightarrow \begin{cases} F'' - KF = 0 \\ \dot{G} - (K+4)G = 0 \end{cases}$

– If $K = 0$, then we get $F'' = 0 \Rightarrow F(x) = Ax + B$, which can only fulfill both BCs if $A = B = 0$ (trivial!) ✗

– If $K > 0$, we get $F(x) = Ae^{\sqrt{K}x} + Be^{-\sqrt{K}x}$ (try yourself to solve that)

→ Let's try to plug in the BCs:

i) $u(0,t)=0 \Leftrightarrow F(0)=0 \Leftrightarrow A\cdot 1 + B\cdot 1 = 0 \Leftrightarrow A+B=0$

ii) $\frac{\partial u}{\partial x}(\pi,t)=0 \Leftrightarrow F'(\pi)=0 \Leftrightarrow \sqrt{k}Ae^{\sqrt{k}\pi} - \sqrt{k}Be^{-\sqrt{k}\pi}=0 \Leftrightarrow \underbrace{\sqrt{k}e^{-\sqrt{k}\pi}}_{\sqrt{k}e^{-\sqrt{k}\pi}}\left(e^{2\sqrt{k}\pi}A-B\right)=\underbrace{0}_{\sqrt{k}e^{-\sqrt{k}\pi}}$

Solution: $A=B=0$ (trivial again!)

— The only possibility left is $k<0$, that is, $k=-\lambda^2$ $(\lambda\in\mathbb{R})$

$\Rightarrow \begin{cases} F''(x) = -\lambda^2 F(x) \\ \dot{G}(t) = (4-\lambda^2)G(t)\end{cases}$ $\Rightarrow$ 1) $F(x) = A\cos\lambda x + B\sin\lambda x$

2) $G(t) = Ce^{(4-\lambda^2)t}$

▸ Boundary conditions:

* $u(0,t)\equiv 0 \Leftrightarrow F(0)=0 \Leftrightarrow A\underbrace{\cos 0}_{1} + B\underbrace{\sin 0}_{0} = 0 \Leftrightarrow \boxed{A=0}$

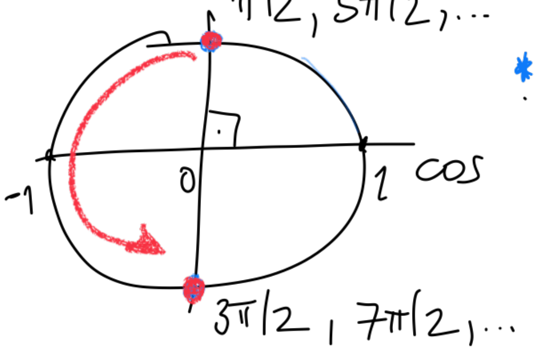
* $\left.\frac{\partial u}{\partial x}\right|_{x=\pi} \equiv 0 \Leftrightarrow F'(\pi)=0 \Leftrightarrow -\lambda \underbrace{A}_{0}\sin\lambda\pi + \lambda B\cos\lambda\pi = 0 \Leftrightarrow B\cos\lambda\pi = 0$

$\boxed{B=0}$ → trivial!

or

$\boxed{\cos\lambda\pi = 0}$

$\Rightarrow \cos\lambda\pi = 0$



* $\cos\lambda\pi = 0$ if $\boxed{\lambda\pi = m\frac{\pi}{2}}$, with m odd

This is the same as writing $\boxed{\lambda\pi = (2n-1)\frac{\pi}{2},\ n\in\mathbb{N}}$

$m = 2n-1$

- We therefore have not only one λ, but infinitely many λ_n:

$$\cancel{\pi}\lambda_n = (2n-1)\frac{\cancel{\pi}}{2} \Rightarrow \boxed{\lambda_n = n - \frac{1}{2}} \longrightarrow$$ for each λ_n, we have a corresponding pair $F_n(x)$, $G_n(t)$.

→ Solution: $$u_n(x,t) = F_n(x)\,G_n(t) = \underbrace{\left[B\sin(\lambda_n x)\right]}_{F_n(x)}\underbrace{\left[C e^{(4-\lambda_n^2)t}\right]}_{G_n(t)} = BC\sin\left(\left(n-\frac{1}{2}\right)x\right)e^{\left[4-\left(n-\frac{1}{2}\right)^2\right]t}$$

- Since C and B are both arbitrary constants, their product $C \cdot B$ is also arbitrary. We can therefore call $B \cdot C$ as simply an arbitrary constant A_n:

$$\boxed{u_n(x,t) = A_n \sin\left(\left(n-\frac{1}{2}\right)x\right)e^{\left[4-\left(n-\frac{1}{2}\right)^2\right]t}, \quad n \in \mathbb{N}}$$

b) Find the solution $u(x,t)$ that, in addition to satisfying the PDE and the BCs, also satisfies the IC:

$$u(x,0) = \sin\left(\frac{3x}{2}\right) + 2\sin\left(\frac{5x}{2}\right) + 3\sin\left(\frac{7x}{2}\right)$$

– Superposition: since we have infinitely many solutions $u_n(x,t)$ and this problem admits superposition (why?), then the sum of solutions $u_n(x,t)$ is also a solution: $u(x,t) = \sum_{n=1}^{\infty} A_n \sin\left(\left(n-\frac{1}{2}\right)x\right) e^{[4-(n-1/2)^2]\,t}$

→ That doesn't mean that all the coefficients A_n will be non-zero! Maybe so, maybe not (depends on the IC).

IC: $u(x,0) = \sin\left(\frac{3x}{2}\right) + 2\sin\left(\frac{5x}{2}\right) + 3\sin\left(\frac{7x}{2}\right)$ (given)

$$u(x,0) = \sum_{n=1}^{\infty} A_n \sin\left(\left(n-\frac{1}{2}\right)x\right)e^{0} = \sum_{n=1}^{\infty} A_n \sin\left(\left(n-\frac{1}{2}\right)x\right)$$ (what we have)

$$\underbrace{A_1}_{\to 0}\sin\left(\underbrace{\left(1-\tfrac{1}{2}\right)}_{1/2}x\right) + A_2\sin\left(\underbrace{\left(2-\tfrac{1}{2}\right)}_{3/2}x\right) + A_3\sin\left(\underbrace{\left(3-\tfrac{1}{2}\right)}_{5/2}x\right) + A_4\sin\left(\underbrace{\left(4-\tfrac{1}{2}\right)}_{7/2}x\right) + \cdots$$

$$= 1\cdot\sin\left(\frac{3x}{2}\right) + 2\sin\left(\frac{5x}{2}\right) + 3\sin\left(\frac{7x}{2}\right)$$

— Now, let's compare these two:

$$\begin{cases} A_1 = 0 \\ A_2 = 1 \\ A_3 = 2 \\ A_4 = 3 \\ A_5 = 0 \\ A_6 = 0 \\ \vdots \end{cases} \Rightarrow \text{only } n = 2, 3, 4 \text{ contribute!}$$

— The solution is therefore:

$$u(x,t) = \overset{A_2}{1}\cdot\sin\left(\frac{3x}{2}\right)e^{[4-(3/2)^2]t} + \overset{A_3}{2}\sin\left(\frac{5x}{2}\right)e^{[4-(5/2)^2]t} + \overset{A_4}{3}\sin\left(\frac{7x}{2}\right)e^{[4-(5/2)^2]t}$$

$$\Rightarrow u(x,t) = e^{7t/4}\sin\left(\frac{3x}{2}\right) + 2e^{-9t/4}\sin\left(\frac{5x}{2}\right) + 3e^{-33t/4}\sin\left(\frac{7x}{2}\right) \quad //$$