Exercise #7

04. October 2022

Problem 1. (Superposition)

The Riccati equation is an ODE that models a variety of physical phenomena. For example, the velocity $v(t)$ of a falling body subjected to gravitational and aerodynamic forces can be described by the initial value problem (IVP)

$$v'(t) - v^2(t) = 1\,, \tag{1}$$
$$v(0) = 0\,. \tag{2}$$

To transform the Riccati equation into a linear, homogeneous ODE, we introduce a new variable $u(t)$ such that $v(t) = -u'(t)/u(t)$ and $u(0) = 1$. This change of variables turns system (1) into a second-order IVP:

$$u''(t) + u(t) = 0\,, \tag{3}$$
$$u'(0) = 0\,, \tag{4}$$
$$u(0) = 1\,, \tag{5}$$

which is easier to solve.

a) Explain why the Riccati equation, in its original form (1), does not admit superposition.

b) Let $u_1(t)$ and $u_2(t)$ be two functions satisfying both Eqs. (3) and (4), and consider also that $u_1(0) = 3$ and $u_2(0) = 2$. Explain why the sum $u_1(t) + u_2(t)$ does *not* satisfy the linear IVP, while the difference $u_1(t) - u_2(t)$ does.

Problem 2. (Classification of PDEs)

Classify the following PDEs into linear/nonlinear, homogeneous/inhomogeneous:

a) Burger's equation:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \frac{\partial^2 u}{\partial x^2} .$$

b) The Laplace equation in polar coordinates ($u = u(r, \theta)$):

$$\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r}\frac{\partial^2 u}{\partial \theta^2} = 0 .$$

c) The Poisson equation describing torsion of prismatic bars:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + 2 = 0 .$$

d) The convection-reaction equation, with a non-zero reaction exponent $n \in \mathbb{N}$:

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} + u^n = 0$$

e) The one-dimensional heat equation with time-dependent heat source

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = \frac{1}{1+t^2} .$$

f) The bi-harmonic equation in two dimensions:

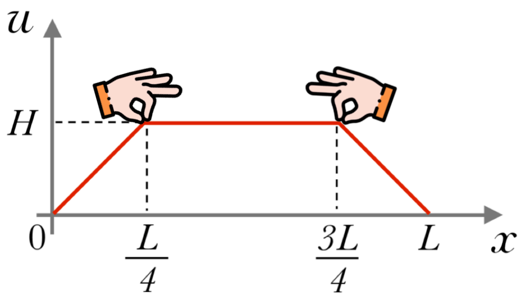
$$\frac{\partial^4 u}{\partial x^4} + 2\frac{\partial^4 u}{\partial x^2 \partial y^2} + \frac{\partial^4 u}{\partial y^4} = 0 .$$

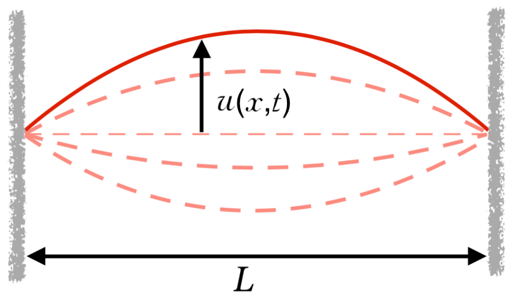
Problem 3. (Wave equation)

The small transversal displacements $u(x, t)$ of a stretched string satisfy the wave equation:

$$\frac{\partial^2 u}{\partial t^2} = \frac{FL}{m}\frac{\partial^2 u}{\partial x^2}, \tag{6}$$

where L and m are the string's length and mass, respectively, and F is the tension pulling the string (all three of them constant). With the string fixed on both sides, we get the boundary conditions $u(0, t) \equiv u(L, t) \equiv 0$. Let's say we grab the string at two points $x = L/4$ and $x = 3L/4$, and pull it up until reaching a height H. Starting from this trapezoidal shape at $t = 0$, the string is released and starts vibrating:





This initial condition for $u(x, t)$ can be written either in three separate linear parts, or also more compactly as

$$u(x, 0) = H\left(\frac{1}{2} + \left|\left|\frac{2x}{L} - 1\right| - 1\right| - \left|\left|\frac{2x}{L} - 1\right| - \frac{1}{2}\right|\right). \tag{7}$$

Moreover, since the string is simply *released*, the velocity $\frac{\partial u}{\partial t}$ will be zero at $t = 0$.

a) Determine the wave speed c in terms of the given constants F, L and m.

b) Determine the vibration period in terms of F, m and L.

c) Using separation of variables, find a general solution $u(x, t)$ for Eq. (6) fulfilling the boundary conditions.

d) Finally, considering $L = 1$, apply the initial conditions to find the solution $u(x, t)$ of this initial-boundary value problem
Hint: if you want, you can use some symbolic integration tool to evaluate the necessary integrals.

The next exercise is optional and should not be handed in!

Problem 4. (Damped wave equation)

One way to include dissipation in wave modelling is via linear damping. In that case, the wave equation becomes

$$\frac{\partial^2 u}{\partial t^2} + 2\gamma\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2}\,,$$

in which c is the wave speed and $\gamma > 0$ is the damping parameter. As for the standard wave equation, we need boundary and initial conditions. Considering the following ones

$$\begin{aligned} u(0,t) &= 0\,, \\ u(L,t) &= 0\,, \\ u(x,0) &= f(x)\,, \\ \left.\frac{\partial u}{\partial t}\right|_{t=0} &= 0 \end{aligned}$$

and assuming $\gamma < c\pi/L$, use the separation of variables technique to show that the solution to this problem is

$$u(x,t) = \sum_{n=1}^{\infty} C_n \mathrm{e}^{-\gamma t}\sin(\lambda_n x)\cos\left(t\sqrt{(c\lambda_n)^2 - \gamma^2}\right)\,, \quad \text{with } \lambda_n := \frac{n\pi}{L}\,,\ n \in \mathbb{N}\,,$$

and determine the constants C_n in terms of $f(x)$.