Exercise #5

16. September 2022

Problem 1. (Trigonometric series)

Determine the coefficients of the trigonometric Fourier series for:

a) $f(x) = \cos^2(x)$,

b) $f(x) = |\sin(x)|$,

c) $f(x) = x^2 - x \;\text{ for }\; -1 \leq x \leq 1$, with period $T = 2L = 2$.

Solution.

a) We see that $f(x)$ is an even function with period π and that we can write

$$f(x) = \cos^2(x) = \frac{1}{2}\left(\cos(2x) + 1\right). \tag{1}$$

Alternative 1:
Since $f(x)$ is already written on the form of a trigonometric Fourier series, we can see that

$$a_0 = \frac{1}{2}, \quad a_1 = \frac{1}{2},$$

and all the other coefficients are 0.
Alternative 2:

Since $f(x)$ is even we have that $b_n = 0$. Calculating a_0 and a_n from the following integrals

$$a_0 = \frac{2}{\pi}\int_0^{\pi/2} \cos^2(x)\mathrm{d}x,$$
$$a_1 = \frac{4}{\pi}\int_0^{\pi/2} \cos^2(x)\cos(2x)\mathrm{d}x,$$
$$a_n = \frac{4}{\pi}\int_0^{\pi/2} \cos^2(x)\cos(2nx)\mathrm{d}x,$$

by using (1) when integrating, gives the same answer as alternative 1.

b) We see that $f(x)$ is a even function with period π, so again $b_n = 0$. Calculating a_0, and using that the function is even:

$$a_0 = \frac{2}{\pi}\int_0^{\pi/2} \sin(x)\mathrm{d}x = \frac{2}{\pi}\left(-\cos\left(\frac{\pi}{2}\right) + \cos(0)\right)\mathrm{d}x = \frac{2}{\pi}$$

Calculating a_1, using that $2\sin(\alpha)\cos(\beta) = \sin(\alpha+\beta) + \sin(\alpha-\beta)$

$$\begin{aligned}
a_n &= \frac{4}{\pi}\int_0^{\pi/2} \sin(x)\cos(2nx)\mathrm{d}x \\
&= \frac{2}{\pi}\int_0^{\pi/2} \left(\sin((1+2n)x) + \sin((1-2n)x)\right)\mathrm{d}x, \\
&= \frac{2}{\pi}\left(\frac{1}{1+2n}\left(-\cos\left((1+2n)\frac{2}{\pi}\right) + \cos(0)\right) + \frac{1}{1-2n}\left(-\cos\left((1-2n)\frac{2}{\pi}\right) + \cos(0)\right)\right) \\
&= \frac{2}{\pi}\left(\frac{1}{1+2n} + \frac{1}{1-2n}\right) \\
&= \frac{4}{\pi(1-4n^2)}
\end{aligned}$$

c) We divide $f(x)$ in two parts, an even function $f_1(x) = x^2$, and an odd function $f_2(x) = -x$. We can now find a_n from f_1, and b_n from f_2.

$$a_0 = \int_0^1 x^2\mathrm{d}x = \frac{1}{3}$$

$$
\begin{aligned}
a_n &= 2\int_0^1 x^2\cos(n\pi x)\mathrm{d}x \\
&= \frac{2}{n\pi}\left[x^2\sin(n\pi x)\right]_0^1 - \frac{4}{n\pi}\int_0^1 x\sin(n\pi x)\mathrm{d}x \\
&= 0 + \frac{4}{n^2\pi^2}\left[x\cos(n\pi x)\right]_0^1 - \frac{4}{n^2\pi^2}\int_0^1 \cos(n\pi x)\mathrm{d}x \\
&= \frac{4}{n^2\pi^2}\cos(n\pi) - \frac{4}{n^3\pi^3}\left[\sin(n\pi x)\right]_0^1 \\
&= \frac{4(-1)^n}{n^2\pi^2}
\end{aligned}
$$

$$
\begin{aligned}
b_n &= -2\int_0^1 x\sin(n\pi x)\mathrm{d}x \\
&= \frac{2}{n\pi}\left[x\cos(n\pi x)\right]_0^1 - \frac{2}{n\pi}\int_0^1 \cos(n\pi x)\mathrm{d}x \\
&= \frac{2cos(n\pi)}{n\pi} - \frac{2}{n^2\pi^2}\left[\sin(n\pi x)\right]_0^1 \\
&= \frac{2(-1)^n}{n\pi}
\end{aligned}
$$

Problem 2. (Half-range trigonometric expansion)

Consider a function $f(x)$ defined for $x \in [0, L]$, not necessarily periodic. To approximate $f(x)$ using Fourier series, we can extend $f(x)$ to the interval $x \in [-L, 0]$ so as to get either an odd or even function. The even extension may be more advantageous in some situations, while in other cases the odd one can be better. We will investigate both scenarios in this exercise, using truncated series.

a) Consider $f(x) = \mathrm{e}^x$, $0 \le x \le 1$. Sketch (draw or plot) the odd and the even extensions of $f(x)$ in the interval $-1 \le x \le 1$.

b) Based on your sketch, answer: which of the extensions should be easier to approximate with a Fourier partial sum? In other words, if we fix the number N of terms in the series, for which case (even or odd extension) can we expect a better approximation of $f(x)$? *Hint:* use your plots to assess the *regularity* (smoothness) of each extended function.

c) Now, consider $f(x) = x - x^2$, $0 \leq x \leq 1$, and sketch again the odd and the even extensions. Which one is smoother now?

d) Still considering the function from (c), compute the Fourier coefficients of its odd and even extensions. In which case do the coefficients (a_n or b_n) go faster to 0 as $n \to \infty$? *Hint:* you may spare a little effort by reusing some of the calculations from Prob. 1 (c).

Solution.

a) See Figure 1.

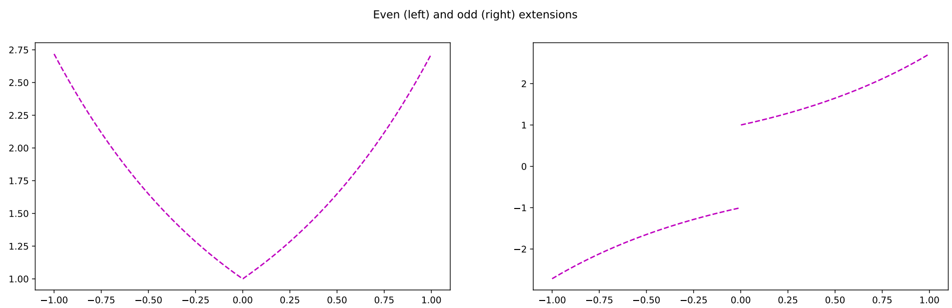

Figure 1: Even and odd extensions for 2a)

b) From the plots we would assume that the even extension is easier to approximate with a Fourier partial sum, since the odd function has a singularity in $x = 0$.

c) See Figure 2. In this case both extensions are continuous, but only the odd extension has a continuous derivative, making it the most smooth.

d) To find the coefficients for the even extension we need to calculate

$$a_0 = \int_0^1 x - x^2 \mathrm{d}x, = \frac{1}{6},$$

$$a_n = 2\int_0^1 (x - x^2)\cos(n\pi x)\mathrm{d}x,$$

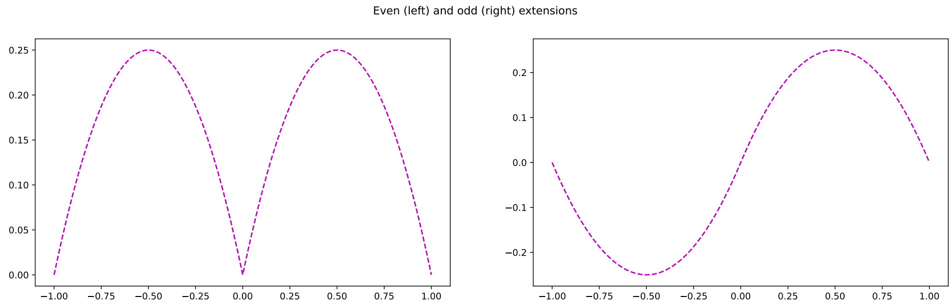

Figure 2: Even and odd extensions for 2c)

and since we can use the answer from 1(c), we only need to compute

$$\begin{aligned} 2\int_0^1 x\cos(n\pi x)\mathrm{d}x &= \frac{2}{n\pi}\left[x\sin(n\pi x)\right]_0^1 - \frac{2}{n\pi}\int_0^1 \sin(n\pi x)\mathrm{d}x \\ &= 0 + \frac{2}{n^2\pi^2}\left[\cos(n\pi x)\right]_0^1 \\ &= \frac{2((-1)^n - 1)}{n^2\pi^2} \end{aligned}$$

Gathering, we have the coefficients for the even extension

$$a_n = \frac{2((-1)^n - 1)}{n^2\pi^2} - \frac{4(-1)^n}{n^2\pi^2} = \frac{-2((-1)^n + 1)}{n^2\pi^2}.$$

For the odd extension, we need to compute

$$b_n = 2\int_0^1 (x - x^2)\sin(n\pi x)\mathrm{d}x,$$

and again we have one of the terms from 1(c), so the new term to compute is

$$\begin{aligned} -2\int_0^1 x^2\sin(n\pi x)\mathrm{d}x, &= \frac{2}{n\pi}\left[x^2\cos(n\pi x)\right]_0^1 - \frac{4}{n\pi}\int_0^1 x\cos(n\pi x)\mathrm{d}x \\ &= \frac{2\cos(n\pi)}{n\pi} - \frac{4}{n^2\pi^2}\left[x\sin(n\pi x)\right]_0^1 + \frac{4}{n^2\pi^2}\int_0^1 \sin(n\pi x)\mathrm{d}x \\ &= \frac{2\cos(n\pi)}{n\pi} - \frac{4}{n^3\pi^3}\left[\cos(n\pi x)\right]_0^1 \\ &= \frac{2(-1)^n}{n\pi} - \frac{4((-1)^n - 1)}{n^3\pi^3} \end{aligned}$$

Gathering, we have that

$$b_n = \frac{2(-1)^n}{n\pi} - \frac{4((-1)^n - 1)}{n^3\pi^3} - \frac{2(-1)^n}{n\pi} = -\frac{4((-1)^n - 1)}{n^3\pi^3}.$$

We now observe that the coefficients for the even part approach 0 as $\frac{1}{n^2}$ and the coefficients for the odd part as $\frac{1}{n^3}$, hence the coefficients goes faster to 0 for the odd extension.

Problem 3. (Complex series)

Determine the coefficients of the complex Fourier series for:

a) $f(x) = \sin^2(x)$,

b) $f(x) = |x-1|$ for $-1 \le x \le 1$, with period $T = 2L = 2$,

c) $f(x) = \mathrm{e}^x$ for $-\pi \le x \le \pi$, with period $T = 2L = 2\pi$,

Solution.

a) We first rewrite $f(x)$ to be expressed by complex functions,

$$f(x) = \sin^2(x) = \frac{2 - e^{2ix} - e^{-2ix}}{4},$$

and note that the period is π. We now have the function on the form of a complex Fourier series, and can see that

$$c_0 = \frac{1}{2}, \quad c_1 = c_{-1} = -\frac{1}{4}.$$

Alternatively, can the coefficients be found by integrating.

b) We first calculate c_0

$$c_0 = \frac{1}{2}\int_{-1}^{1} |x-1| \mathrm{d}x = \frac{1}{2}\int_{-1}^{1} (1-x) \mathrm{d}x = 1.$$

Next, for $n \neq 0$, we need to compute the following integral

$$c_n = \frac{1}{2}\int_{-1}^{1} |x-1| e^{-in\pi x} \mathrm{d}x = \int_{-1}^{1} (1-x) e^{-in\pi x} \mathrm{d}x,$$

which we separate into two parts,

$$\int_{-1}^{1} e^{-in\pi x} \mathrm{d}x = \frac{1}{-in\pi}\left(e^{-in\pi} - e^{in\pi}\right) = 0,$$

$$-\int_{-1}^{1} x e^{-in\pi x} \mathrm{d}x = \frac{1}{in\pi}\left(e^{-in\pi} + e^{in\pi}\right) - \frac{1}{in\pi}\int_{-1}^{1} e^{-in\pi x} \mathrm{d}x = \frac{-2i}{n\pi}(-1)^n,$$

$$c_n = \frac{-i}{n\pi}(-1)^n.$$

c)

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{e}^x \mathrm{e}^{-inx}\, \mathrm{d}x = \left[-\frac{\mathrm{e}^{(1-in)x}}{2\pi(1-in)}\right]_{-\pi}^{\pi} = \frac{\mathrm{e}^{-i\pi n}(\mathrm{e}^{\pi} - \mathrm{e}^{-\pi})}{2\pi(1+in)} = \frac{\sinh \pi}{\pi}\,\frac{(-1)^n(1+in)}{1+n^2}.$$

Problem 4. (Parseval's theorem)

Consider two periodic functions $f(x)$ and $g(x)$ and their truncated Fourier series $f_N(x)$ and $g_N(x)$. The errors E_N between each function and its trigonometric approximation can be computed using Parseval's identity.

a) Let $f(x) = \mathrm{e}^x$ for $-\pi \le x \le \pi$, with period $T = 2L = 2\pi$. Based on the Fourier coefficients of $f(x)$, calculate the error E_N for $N = 2$, 4 and 8.
Hint: you have already computed the (complex) coefficients for this function on for Problem 3 (c).

b) Now, do the same for $g(x) = \mathrm{e}^{|x|}$, $-\pi \le x \le \pi$, with period $T = 2L = 2\pi$.

c) Why is the error so much larger when approximating $f(x)$, than in the case of $g(x)$?

Solution.

From Parseval's identity, we have that

$$E_N = \int_{-\pi}^{\pi} f^2(x)\,\mathrm{d}x - 2\pi \sum_{n=-N}^{N} |c_n|^2 .$$

a) We first integrate

$$\int_{-\pi}^{\pi} f^2(x)\mathrm{d}x = \int_{-\pi}^{\pi} e^{2x}\mathrm{d}x = \frac{e^{2\pi} - e^{-2\pi}}{2} = \sinh(2\pi)$$

$$E_N = \sinh(2\pi) - 2\pi \sum_{n=-N}^{N} \frac{\sinh^2(\pi)}{\pi^2(1+n^2)} = \sinh(2\pi) - \sum_{n=-N}^{N} \frac{2\sinh^2(\pi)}{\pi(1+n^2)}$$

Inserting values for N: $E_2 = 64, E_4 = 37, E_8 = 19,9$

b) We first find the Fourier coefficients for $g(x)$

$$\begin{aligned}
c_n &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{e}^{|x|}\mathrm{e}^{-inx}\,\mathrm{d}x = \frac{1}{2\pi}\int_{-\pi}^{0} \mathrm{e}^{-x}\mathrm{e}^{-inx}\,\mathrm{d}x + \frac{1}{2\pi}\int_{0}^{\pi} \mathrm{e}^{x}\mathrm{e}^{-inx}\,\mathrm{d}x \\
&= \frac{1}{2\pi}\left(\frac{-1}{1+in}\left(1 - e^{(1+in)\pi}\right) + \frac{1}{1-in}\left(e^{(1-in)\pi} - 1\right)\right) \\
&= \frac{1}{2\pi}\frac{1}{1+n^2}\left(-2 + \mathrm{e}^{\pi}\mathrm{e}^{-in\pi} + \mathrm{e}^{\pi}\mathrm{e}^{in\pi}\right) \\
&= \frac{1}{\pi(1+n^2)}\left(-1 + \mathrm{e}^{\pi}(-1)^n\right) .
\end{aligned}$$

As in a), we need to compute the integral of g(x) squared

$$\int_{-\pi}^{\pi} g^2(x)\mathrm{d}x = \int_{-\pi}^{\pi} e^{2|x|}\mathrm{d}x = e^{2\pi} - 1$$

$$E_N = e^{2\pi} - 1 - 2\pi \sum_{n=-N}^{N} \left(\frac{1}{\pi(1+n^2)}\left(-1 + \mathrm{e}^{\pi}(-1)^n\right)\right)^2$$

Inserting values for N gives: $E_2 = 12, E_4 = 2.4, E_8 = 0.4$

c) g(x) is smoother than f(x) and the Fourier series are therefore a better approximation.

The next exercises are optional and should not be handed in!

Problem 5. (Infinite sums)

In this exercise, we will use Fourier series and coefficients to show a few identities.

a) Consider the 2π-periodic function $f(x) = \begin{cases} -\pi - x & \text{if } -\pi < x < -\frac{\pi}{2}, \\ x & \text{if } -\frac{\pi}{2} < x < \frac{\pi}{2}, \\ \pi - x & \text{if } \frac{\pi}{2} < x \leq \pi. \end{cases}$
Compute its Fourier coefficients and use the result, together with Parseval's identity, to show that

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^4} = \frac{\pi^4}{96}.$$

b) Consider the function $f(x) = x$, defined for $x \in [0, 1]$. Compute the Fourier series of its *even extension*, and use the result to show that

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \ldots = \frac{\pi^2}{8}.$$

c) Now, considering the same function $f(x)$ from (b), compute the Fourier series of its *odd extension* and use the result to show that

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \ldots = \frac{\pi}{4}.$$

Solution.

a) We see that this is an odd function, meaning that $a_n = 0$.

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin(nx) \mathrm{d}x = \frac{2}{\pi} \left(\int_0^{\pi/2} x \sin(nx) \mathrm{d}x + \int_{\pi/2}^{\pi} (\pi - x) \sin(nx) \mathrm{d}x \right)$$
$$= \begin{cases} \frac{4(-1)^{n+1}}{\pi n^2} & \text{if n is odd} \\ 0 & \text{if n is even} \end{cases}$$

Now integrating f^2

$$\int_0^\pi f^2(x)\mathrm{d}x = \frac{\pi^3}{6} \tag{2}$$

and using Parseval's identity gives,

$$\frac{\pi^3}{6} - \sum_{n=1}^{N} \frac{16}{\pi(2n+1)^4} = 0$$
$$\sum_{n=1}^{N} \frac{1}{(2n+1)^4} = \frac{\pi^4}{96}$$

b) If we reuse our computations from 2d) we have that

$$a_n = \frac{2((-1)^n - 1)}{n^2\pi^2} = \begin{cases} \frac{-4}{\pi n^2} & \text{if n is odd} \\ 0 & \text{if n is even} \end{cases}$$

In addition, we need to compute a_0,

$$a_0 = \int_0^1 x\mathrm{d}x = \frac{1}{2}$$

The Fourier series for the function can therefore be written as

$$f(x) = \frac{1}{2} - \frac{4}{\pi^2}\left(\frac{\cos(\pi x)}{1^2} + \frac{3\cos(\pi x)}{3^2} + \frac{5\cos(\pi x)}{5^2} + \dots\right)$$

and by setting $x = 0$ we have that

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \dots = \frac{\pi^2}{8}$$

c) Now, we can reuse our computations from 1c)

$$b_n = \frac{2(-1)^{n+1}}{n\pi},$$

and the Fourier series can be written as

$$f(x) = \frac{2}{\pi}\left(\frac{\sin(\pi x)}{1} - \frac{\sin(2\pi x)}{2} + \frac{\sin(3\pi x)}{3} - \frac{\sin(4\pi x)}{4} + \frac{\sin(5\pi x)}{5} \dots\right).$$

By setting $x = \frac{\pi}{2}$ we have that

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \dots = \frac{\pi}{4}.$$