Exercise #5

16. September 2022

Problem 1. (Trigonometric series)

Determine the coefficients of the trigonometric Fourier series for:

a) $f(x) = \cos^2(x)$,

b) $f(x) = |\sin(x)|$,

c) $f(x) = x^2 - x$ for $-1 \le x \le 1$, with period $T = 2L = 2$.

Problem 2. (Half-range trigonometric expansion)

Consider a function $f(x)$ defined for $x \in [0, L]$, not necessarily periodic. To approximate $f(x)$ using Fourier series, we can extend $f(x)$ to the interval $x \in [-L, 0]$ so as to get either an odd or even function. The even extension may be more advantageous in some situations, while in other cases the odd one can be better. We will investigate both scenarios in this exercise, using truncated series.

a) Consider $f(x) = e^x$, $0 \le x \le 1$. Sketch (draw or plot) the odd and the even extensions of $f(x)$ in the interval $-1 \le x \le 1$.

b) Based on your sketch, answer: which of the extensions should be easier to approximate with a Fourier partial sum? In other words, if we fix the number N of terms in the series, for which case (even or odd extension) can we expect a better approximation of $f(x)$? *Hint:* use your plots to assess the *regularity* (smoothness) of each extended function.

c) Now, consider $f(x) = x - x^2$, $0 \leq x \leq 1$, and sketch again the odd and the even extensions. Which one is smoother now?

d) Still considering the function from (c), compute the Fourier coefficients of its odd and even extensions. In which case do the coefficients (a_n or b_n) go faster to 0 as $n \to \infty$?
Hint: you may spare a little effort by reusing some of the calculations from Prob. 1 (c).

Problem 3. (Complex series)

Determine the coefficients of the complex Fourier series for:

a) $f(x) = \sin^2(x)$,

b) $f(x) = |x - 1|$ for $-1 \leq x \leq 1$, with period $T = 2L = 2$,

c) $f(x) = e^x$ for $-\pi \leq x \leq \pi$, with period $T = 2L = 2\pi$,

Problem 4. (Parseval's theorem)

Consider two periodic functions $f(x)$ and $g(x)$ and their truncated Fourier series $f_N(x)$ and $g_N(x)$. The errors E_N between each function and its trigonometric approximation can be computed using Parseval's identity.

a) Let $f(x) = e^x$ for $-\pi \leq x \leq \pi$, with period $T = 2L = 2\pi$. Based on the Fourier coefficients of $f(x)$, calculate the error E_N for $N = 2$, 4 and 8.
Hint: you have already computed the (complex) coefficients for this function on for Problem 3 (c).

b) Now, do the same for $g(x) = e^{|x|}$, $-\pi \leq x \leq \pi$, with period $T = 2L = 2\pi$.

c) Why is the error so much larger when approximating $f(x)$, than in the case of $g(x)$?

The next exercises are optional and should not be handed in!

Problem 5. (Infinite sums)

In this exercise, we will use Fourier series and coefficients to show a few identities.

a) Consider the 2π-periodic function $f(x) = \begin{cases} -\pi - x & \text{if } -\pi < x < -\frac{\pi}{2}, \\ x & \text{if } -\frac{\pi}{2} < x < \frac{\pi}{2}, \\ \pi - x & \text{if } \frac{\pi}{2} < x \leq \pi. \end{cases}$

Compute its Fourier coefficients and use the result, together with Parseval's identity, to show that

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^4} = \frac{\pi^4}{96}.$$

b) Consider the function $f(x) = x$, defined for $x \in [0, 1]$. Compute the Fourier series of its *even extension*, and use the result to show that

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \ldots = \frac{\pi^2}{8}.$$

c) Now, considering the same function $f(x)$ from (b), compute the Fourier series of its *odd extension* and use the result to show that

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \ldots = \frac{\pi}{4}.$$