Lecture 22 — **Fourier Transform**

## Complex Form of the Fourier integral:

- (Real) Fourier integral:

$$f(x) = \int_0^{\infty} \big(A(\omega)\cos \omega x + B(\omega)\sin \omega x\big)\, d\omega$$

with
$$A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos \omega v \, dv$$

$$B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\sin \omega v \, dv .$$

$$\Rightarrow f(x) = \frac{1}{\pi}\int_0^{\infty}\int_{-\infty}^{\infty} f(v)\underbrace{\big(\cos\omega v \cdot \cos \omega x + \sin \omega v \sin \omega x\big)}\, dv\, d\omega$$

$= \cos(\omega v - \omega x)$ (addition formula for cosine)

$= \cos(\omega x - \omega v)$ (cosine is even

$$= \frac{1}{\pi}\int_0^{\infty}\underbrace{\left[\int_{-\infty}^{\infty} f(v)\cdot\cos(\omega x - \omega v)\, dv\right]}\, d\omega$$

even function of $\omega$, because the cosine is even and $f$ does not depend on $\omega$

We have: $\int_{-a}^{a}$ even function $= 2\cdot\int_0^a$ even fct

$$\Rightarrow \frac{1}{2}\cdot\int_{-a}^{a} \text{even function} = \int_0^a \text{even funct}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(v)\cos(\omega x - \omega v)\, dv\, d\omega \quad + \quad 0$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(v)\cos(\omega x - \omega v)\, dv\, d\omega$$

$= 0$

$$+\frac{1}{2\pi}\int_{-\infty}\underbrace{\int_{-\infty} f(v)\sin(\omega x - \omega v)\,dv}_{\text{is an odd function of } \omega\text{, as sine is odd and } f \text{ does not depend on } \omega}\,d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(v)\left(\cos(\omega x - \omega v) + \sin(\omega x - \omega v)\right)dv\,d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(v)\left(\cos(\omega x - \omega v) + \underbrace{i}_{=\sqrt{-1}}\cdot\sin(\omega x - \omega v)\right)dv\,d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(v)\cdot e^{i\omega(x-v)}\,dv\,d\omega \qquad (1)$$

## Fourier Transform

From (1), we obtain

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\underbrace{\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(v)\,e^{-i\omega v}\,dv\right)}_{=:\ \hat{f}(\omega)\ =\ \text{Fourier transform of } f} e^{i\omega x}\,d\omega$$

Definition: We call

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\,e^{-i\omega x}\,dx \qquad \left(= \mathcal{F}(f)\right)$$

the Fourier transform of $f$.

The inverse Fourier transform of $\hat{f}(\omega)$ is

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{f}(\omega)\,e^{i\omega x}\,d\omega.$$

<u>Theorem</u> : Existence of the Fourie Transform

If $f(x)$ is <u>absolutely integrable</u> on the x-axis and piecewise continuous on every finite interval, then the Fourie transform $\hat{f}(\omega)$ of $f(x)$ exists.

<u>Example</u> : $f(x) = \begin{cases} 1 & : \ |x| < 1 \\ 0 & : \ \text{else} \end{cases}$

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-i\omega x}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} 1 \cdot e^{-i\omega x}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \left[ -\frac{1}{i\omega} \cdot e^{-i\omega x} \right]_{-1}^{1}$$

$$= \frac{1}{\sqrt{2\pi}} \left( -\frac{1}{i\omega} \left( e^{-i\omega} - e^{i\omega} \right) \right)$$

We have : $\underline{e^{i\omega}} = \cos\omega + i \sin\omega$

$\underline{e^{-i\omega}} = \cos\omega - i \sin\omega$.

And thus :

$$\underline{\underline{\hat{f}(\omega)}} = \frac{1}{\sqrt{2\pi}} \cdot \left( -\frac{1}{i\omega} \right) \cdot \left( \cancel{\cos\omega} - i\sin\omega - \left( \cancel{\cos\omega} + i \sin\omega \right) \right)$$

$$= \frac{2 \cancel{i} \sin\omega}{\sqrt{2\pi}\, \cancel{i} \omega}$$

$$= \frac{2}{\sqrt{2\pi}} \cdot \frac{\sin\omega}{\omega}$$

**Example**: let $f(x) = \begin{cases} e^{-ax} & : x > 0 \\ 0 & : x < 0 \end{cases}$, $a > 0$.

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-ax} \cdot e^{-i\omega x} \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{x(-a-i\omega)} \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[ \frac{1}{-a-i\omega} \cdot e^{x(-a-i\omega)} \right]_0^\infty$$

$$= \frac{1}{\sqrt{2\pi} \cdot (-a-i\omega)} \cdot \Big( \underbrace{\lim_{T \to \infty} e^{T(-a-i\omega)}}_{e^{T(-a-i\omega)} = e^{-T(a+i\omega)} \to 0 \text{ as } a > 0 \text{ and } |e^{-i\omega T}| = 1} - \underbrace{e^0}_{=1} \Big)$$

$$= \frac{1}{\sqrt{2\pi}\,(a+i\omega)}$$

## Linearity

**Theorem**: Linearity of the Fourier Transform
The Fourier transform is a <u>linear</u> operator, that is, for any functions $f(x)$ and $g(x)$ whose Fourier transforms exist, and any constants $a, b$, the Fourier transform of $af + bg$ exists, and

$$\mathcal{F}(af + bg) = a\mathcal{F}(f) + b\mathcal{F}(g)$$

$$\mathcal{F}(af + bg) = a\,\mathcal{F}(f) + b\,\mathcal{F}(g).$$

Proof: $\mathcal{F}(af+bg) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(af(x) + bg(x)\right) \cdot e^{-i\omega x}\, dx$

$$= \frac{1}{\sqrt{2\pi}} \cdot a \int_{-\infty}^{\infty} f(x) e^{-i\omega x}\, dx + \frac{1}{\sqrt{2\pi}} \cdot b \int_{-\infty}^{\infty} g(x) e^{-i\omega x}\, dx$$

$$= a \cdot \mathcal{F}(f) + b \cdot \mathcal{F}(g). \qquad \blacksquare$$

Fourie Transform of Derivatives

Theorem: Fourie Transform of the Derivative of $f(x)$

Let $f(x)$ be continuous on the x-axis and

$$f(x) \to 0 \quad \text{as} \quad |x| \to \infty.$$

Furthermore, let $f'(x)$ be absolutely integrable on the x-axis. Then

$$\mathcal{F}(f'(x)) = i\omega\, \mathcal{F}(f(x)).$$

Proof: Due to the definition of the Fourie transform, we have:

$$\mathcal{F}(f'(x)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f'(x) \cdot e^{-i\omega x}\, dx$$

Integration by parts

$$\overset{\downarrow}{=} \frac{1}{\sqrt{2\pi}} \left( \left[ f(x) \cdot e^{-i\omega x} \right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f(x)(-i\omega) e^{-i\omega x}\, dx \right.$$

$$= \frac{1}{\sqrt{2\pi}} \left( 0 - 0 + i\omega \int_{-\infty}^{\infty} f(x) e^{-i\omega x}\, dx \right)$$

as $f(x) \to 0$ as $|x| \to \infty$

$$= i\omega \cdot \underbrace{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \cdot e^{-i\omega x} \, dx}_{= \hat{f}(\omega) = \mathcal{F}(f(x))}$$

$$= i\omega \cdot \mathcal{F}(f) \qquad \blacksquare$$

Remark: We have

$$\mathcal{F}(f''(x)) = i\omega \cdot \mathcal{F}(f'(x))$$
$$= i\omega \cdot i\omega \, \mathcal{F}(f)$$
$$= \underbrace{i^2}_{=-1} \omega^2 \, \mathcal{F}(f)$$
$$= -\omega^2 \, \mathcal{F}(f)$$

Convolution

Definition: The convolution $f * g$ of two functio[ns] $f$ and $g$ is defined by

$$h(x) := (f * g)(x) := \int_{-\infty}^{\infty} f(p) \, g(x-p) \, dp$$
$$= \int_{-\infty}^{\infty} g(p) \, f(x-p) \, dp.$$

Theorem: Convolution Theorem

Suppose that $f(x)$ and $g(x)$ are piecewise continuous, bounded, and absolutely integrable on the x-axis. Then

$$\boxed{\mathcal{F}(f * g) = \sqrt{2\pi}\, \mathcal{F}(f) \cdot \mathcal{F}(g).}$$

Proof: We have

$$\mathcal{F}(f*g) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} f(p) g(x-p)\, dp \right) \cdot e^{-i\omega x}\, dx$$

change of order of integration

$$\overset{\downarrow}{=} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(p) g(x-p)\, e^{-i\omega x}\, dx\, dp$$

substitution: $q = x - p$

$\Rightarrow x = q + p, \quad \frac{dx}{dq} = 1$

$\Rightarrow dx = dq$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(p)\, g(q)\, e^{-i\omega(q+p)}\, dq\, dp$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(p) \cdot e^{-i\omega p}\, dp \cdot \int_{-\infty}^{\infty} g(q)\, e^{-i\omega q}\, dq$$

$$= \frac{1}{\cancel{\sqrt{2\pi}}} \cdot \cancel{\sqrt{2\pi}}\, \mathcal{F}(f) \cdot \sqrt{2\pi} \cdot \mathcal{F}(g)$$

$$= \sqrt{2\pi}\, \mathcal{F}(f) \cdot \mathcal{F}(g). \qquad \blacksquare$$

## Discrete Fourier Transform

- often a function $f(x)$ is only given in terms of values at finitely many points (samples)
- let $f(x)$ be periodic with period $2\pi$
- We assume that N measurements of $f(x)$ are

taken on $[0, 2\pi]$ at regularly spaced points

$$\boxed{x_k = \frac{2\pi k}{N}}, \quad k = 0, 1, 2, \dots, N-1.$$

( f is "sampled" at these points).

- We now want to determine a <u>complex</u> <u>trigonometric polynomial</u>

$$q(x) = \sum_{n=0}^{N-1} c_n e^{inx}$$

that <u>interpolates</u> $f(x)$ at the nodes $x_k$:

$$f(x_k) =: f_k = q(x_k).$$

$$\Rightarrow \quad f_k = \sum_{n=0}^{N-1} c_n e^{inx_k} \quad \Big| \text{ multiply with } e^{-imx_k} \text{ and sum over } k \text{ from } 0 \text{ to } N-1$$

$$\Rightarrow \sum_{k=0}^{N-1} f_k \cdot e^{-imx_k} = \sum_{k=0}^{N-1} \sum_{n=0}^{N-1} c_n \underbrace{e^{inx_k} \cdot e^{-imx_k}}_{= e^{i(n-m)x_k}}$$

$$= \sum_{n=0}^{N-1} c_n \cdot \sum_{k=0}^{N-1} e^{i(n-m)\frac{2\pi k}{N}}$$

We have:

$$e^{i(n-m)\frac{2\pi k}{N}} = \Big(\overbrace{e^{i(n-m)\frac{2\pi}{N}}}^{=r}\Big)^k$$

- If $n = m$, then $e^0 = 1$, and $\sum_{k=0}^{N-1} e^0 = \boxed{N}$
- If $n \neq m$, then $\sum_{k=0}^{N-1} r^k = \frac{1 - r^N}{1 - r} = 0$

because $\boxed{r^N = 1}$ :

$$\left(e^{i(n-m)\frac{2\pi}{N}}\right)^N = e^{i(n-m)2\pi}$$

$$= \underbrace{\cos(2\pi(n-m))}_{=1} + i \underbrace{\sin 2\pi (n}_{=0}$$

$$= 1$$