Lecture 21

Theorem: If $f(x)$ is piecewise continuous in every finite interval and has a right-hand ~~side~~ derivative and a left-hand derivative at every point and if

$$\lim_{a\to\infty}\int_{-a}^{0}|f(x)|\,dx + \lim_{b\to\infty}\int_{0}^{b}|f(x)|\,dx$$

exists, then $f(x)$ can be represented by a Fourier integral.
At a point where $f(x)$ is discontinuous, the value of the Fourier integral equals the average of the left- and right-hand limits of $f(x)$ at that point.

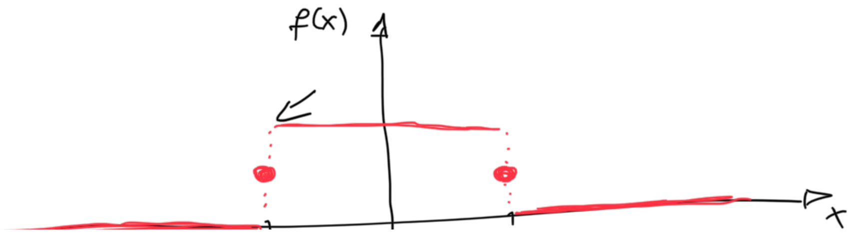
Proof: See A. Zygmund, R. Fefferman: Trigonometric series. 3rd edition. Cambridge University Press. New York. 2002.

Application of Fourier integrals

- Fourier integrals are used to solve ODEs and PDEs
- also used in integration.

Example: Find the Fourier integral representation of the function:

$$f(x) = \begin{cases} 1 & : \quad |x| < 1 \\ 0 & : \quad |x| > 1 \end{cases}$$



We get:

$$\underline{\underline{A(\omega)}} = \frac{1}{\pi} \int_{-\infty}^{\infty} f(v) \cdot \cos \omega v \, dv$$

$$= \frac{1}{\pi} \int_{-1}^{1} 1 \cdot \cos \omega v \, dv$$

$$= \frac{1}{\pi} \cdot \frac{1}{\omega} \cdot \Big[\sin \omega v \Big]_{-1}^{1}$$

$$= \frac{1}{\pi \omega} \cdot \Big(\sin \omega - \underbrace{\sin(-\omega)}_{= -\sin(\omega)} \Big)$$

$$= \underline{\underline{\frac{1}{\pi \omega} \cdot 2 \sin \omega}}$$

$$\underline{\underline{B(\omega)}} = \frac{1}{\pi} \int_{-\infty}^{\infty} f(v) \sin \omega v \, dv$$

$$= \frac{1}{\pi} \int_{-1}^{1} 1 \cdot \sin \omega v \, dv$$

$$= \frac{1}{\pi} \cdot \frac{1}{\omega} \Big[-\cos \omega v \Big]_{-1}^{1}$$

$$= \frac{1}{\pi \omega} \Big(-\cos \omega + \underbrace{\cos(-\omega)}_{= \cos \omega} \Big)$$

$$= \underline{\underline{0}}$$

We get: except at the points of discontinuity of .

$$\underbrace{f(x)}\ \text{"="} \int_{0}^{\infty} \frac{1}{\pi \omega} \cdot 2 \sin \omega \cdot \cos \omega x \, d\omega ,$$

given functions

Fourier integral of f

$$= \frac{2}{\pi} \int_0^\infty \frac{\sin w \cos wx}{w}\, dw$$

Def. of f and preceding theorem

$$= \begin{cases} 1 & : \quad |x| < 1 \\ 0 & : \quad |x| > 1 \\ 1/2 & : \quad |x| = 1 \end{cases}$$

$\Rightarrow$ value of the Fourier integral at points of discontinuity of f equals the average of the left- and right-hand limit: $\frac{0+1}{2} = \frac{1}{2}$

This yields:

$$\underbrace{\int_0^\infty \frac{\sin w \cos wx}{w}\, dw}_{\text{"Dirichlet's discontinuous factor"}} = \begin{cases} \frac{\pi}{2} & : \quad |x| < 1 \\ 0 & : \quad |x| > 1 \\ \frac{\pi}{4} & : \quad |x| = 1 \end{cases}$$

For $x = 0$, we obtain:

$$\int_0^\infty \frac{\sin w \cdot \overbrace{\cos w \cdot 0}^{=1}}{w}\, dw = \int_0^\infty \frac{\sin w}{w}\, dw = \frac{\pi}{2}$$

This is the limit of the so-called sine-integral

$$\operatorname{Si}(u) := \boxed{\int_0^u \frac{\sin w}{w}\, dw}$$

as $u \to \infty$.

Approximation:

Fourier series

$$f(x) \approx a_0 + \sum_{n=1}^{N} (a_n \cos nx + b_n \sin nx)$$

Fourier integral

$$f(x) \approx \int_0^{a} (A_a(\omega) \cos \omega x + B_a(\omega) \sin \omega x)\, d\overset{w}{\cancel{x}}$$

with $$A_a(\omega) = \frac{1}{\pi} \int_{-a}^{a} f(v) \cos \omega v \, dv$$

$$B_a(\omega) = \frac{1}{\pi} \int_{-a}^{a} f(v) \sin \omega v \, dv.$$

In the above example, we therefore have from (*):

$$f(x) \approx \frac{2}{\pi} \int_0^{a} \frac{\sin \omega \cos \omega x}{\omega}\, d\omega.$$

Gibbs-Phenomenon

- Oscillations of the Fourier integral near the points of discontinuity of $f(x)$
- When $a \to \infty$, the oscillations are shifted closer to the points of discontinuity (they do <u>not</u> disappear as $a \to \infty$)

Explanation by means of the preceding example:

$$f(x) \approx \frac{2}{\pi} \int_0^{a} \frac{\cos \omega x \sin \omega}{\omega}\, d\omega$$

$$\cos x \sin y = \frac{1}{2} (\sin(x+y) + \sin(y-x))$$

$$= 1 \int^{\boxed{a}} \sin(\overbrace{\omega x + \omega}^{=:t})\, d\omega + 1 \int^{a} \sin(\omega - \omega x)\, d\omega$$

$$\pi \int_0 \frac{\ldots}{\omega}\, d\omega \qquad \pi \int_0 \frac{\ldots}{\omega}\, d\omega$$

Substitution:

$$t := \omega x + \omega = \omega(1+x)$$

$$\Rightarrow \omega = \frac{t}{1+x} \Rightarrow \frac{1}{\omega} = \frac{1+x}{t}$$

$$\Rightarrow \frac{d\omega}{dt} = \frac{1}{1+x}$$

$$\Rightarrow d\omega = \frac{dt}{1+x}$$

Substitution:

$$-t = \omega - \omega x = \omega(1-x)$$

$$\Rightarrow \omega = -\frac{t}{1-x} \Rightarrow \frac{1}{\omega} = -\frac{1-x}{t}$$

$$\Rightarrow \frac{d\omega}{dt} = -\frac{1}{1-x}$$

$$\Rightarrow d\omega = -\frac{dt}{1-x}$$

$$= \frac{1}{\pi} \int_0^{a(x+1)} \frac{\sin t}{t} \cdot \cancel{(1+x)} \cdot \frac{dt}{1+\cancel{x}} + \frac{1}{\pi} \int_0^{a(x-1)} -\frac{\sin(-t)}{t} \cdot \cancel{(1-x)} \cdot \frac{(-dt)}{\cancel{1-x}}$$

$$= \frac{1}{\pi} \int_0^{a(x+1)} \frac{\sin t}{t}\, dt + \frac{1}{\pi} \int_0^{a(x-1)} \frac{\overbrace{\sin(-t)}^{=-\sin(t)}}{t}\, dt$$

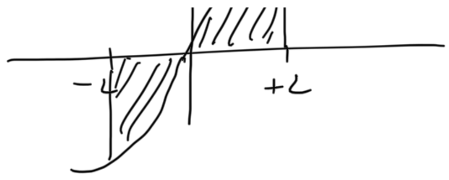
$$= \frac{1}{\pi} \int_0^{a(x+1)} \frac{\sin t}{t}\, dt - \frac{1}{\pi} \int_0^{a(x-1)} \frac{\sin t}{t}\, dt$$

$$= \frac{1}{\pi} \cdot Si\big(a(x+1)\big) - \frac{1}{\pi} Si\big(a(x-1)\big)$$

Fourier Cosine integral and Fourier Sine integral

- If f has a Fourier integral representation and if f is even, then

$$B(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} \underbrace{\underbrace{f(v)}_{\text{even}} \underbrace{\sin \omega v}_{\text{odd}}}_{\text{odd}}\, dv = 0$$



Then, in this case,

$$\boxed{f(x) = \int_0^\infty A(w) \cos wx \, dw}$$ (Fourier cosine integral)

with $A(w) = \frac{2}{\pi} \int_0^\infty \underbrace{\underbrace{f(v)}_{\text{even}} . \underbrace{\cos wv}_{\text{even}}}_{\text{even}} \, dv$

(as $\frac{1}{\pi} \int_{-\infty}^{\infty}$ even function $= \frac{2}{\pi} \int_0^\infty$ even function)

- If f has a Fourier integral representation and is odd, then

$$A(w) = \frac{1}{\pi} \int_{-\infty}^{\infty} \underbrace{\underbrace{f(v)}_{\text{odd}} \underbrace{\cos wv}_{\text{even}}}_{\text{odd}} \, dv = 0.$$

Thus, in this case:

$$\boxed{f(x) = \int_0^\infty B(w) \sin wx \, dw}$$ Fourier sine integral

with $B(w) = \frac{2}{\pi} \int_0^\infty f(v) \sin wv \, dv$.