Lecture 20

- Third integral:

$$\int_{-\pi}^{\pi} F^2\,dx = \int_{-\pi}^{\pi} (A_0 + A_1 \cos x + B_1 \sin x + \dots + A_N \cos Nx + B_N \sin Nx)^2\,dx$$

$$= \pi\,(2A_0^2 + A_1^2 + \dots + A_N^2 + B_1^2 + \dots + B_N^2)$$

We obtain for the error:

$$E = \int_{-\pi}^{\pi} f^2\,dx - 2\pi\left(2A_0 a_0 + \sum_{n=1}^{N}(A_n a_n + B_n b_n)\right) + \pi\left(2A_0^2 + \sum_{n=1}^{N}(A_n^2 + B_n^2)\right)$$

If we take $a_n = A_n$ and $B_n = b_n$ for all n, then we get:

$$E^* := \int_{-\pi}^{\pi} f^2(x)\,dx - \pi\left(2a_0^2 + \sum_{n=1}^{N}(a_n^2 + b_n^2)\right).$$

Moreover, we have

$$E - E^* = -2\pi\left(2A_0 a_0 + \sum_{n=1}^{N} A_n a_n + B_n b_n\right) + \pi\left(2A_0^2 + \sum_{n=1}^{N}(A_n^2 + B_n^2)\right) - \left(-\pi\left(2a_0^2 + \sum_{n=1}^{N}(a_n^2 + b_n^2)\right)\right)$$

$$= \pi\left(2\underbrace{(A_0 - a_0)^2}_{\geq 0} + \sum_{n=1}^{N}\left(\underbrace{(A_n - a_n)^2}_{\geq 0} + \underbrace{(B_n - b_n)^2}_{\geq 0}\right)\right)$$

$$\geq 0$$

$$\Longrightarrow E \geq E^* \quad \text{and} \quad E = E^* \iff A_0 = a_0,$$
$$A_1 = a_1,\ B_1 = b_1,\ \ldots,\ B_N = b_N .$$

Thus, we have the following theorem:

Theorem (Minimum Square Error):

The square error of $F = A_0 + \sum_{n=1}^{N \cancel{\infty}} (A_n \cos nx + B_n \sin nx)$ relative to f on the interval $[-\pi, \pi]$ is minimum if and only if the coefficients of F are the Fourier coefficients of f.

The minimum value of the error is

$$E^* := \underline{\int_{-\pi}^{\pi} f^2\,dx} - \pi\Big(2a_0^2 + \sum_{n=1}^{N} \underline{(a_n^2 + b_n^2)}\Big). \qquad (1$$

Remark: By (1), it can be seen that E^* cannot increase as N increases, but it may decrease.

Thus: With increasing N, the partial sums of the Fourier series of a function f yield better and better approximations of f.

Remark: Bessel's Inequality:

$$E^* \overset{(1)}{=} \int_{-\pi}^{\pi} f^2\,dx - \pi\Big(2a_0^2 + \sum_{n=1}^{N}(a_n^2 + b_n^2)\Big) \geq 0$$

$$\Longrightarrow \frac{1}{\pi}\int_{-\pi}^{\pi} f^2\,dx \geq 2a_0^2 + \sum_{n=1}^{N}(a_n^2 + b_n^2)$$

Since (1) holds for all N, we get:

$$\frac{1}{\pi}\underbrace{\int_{-\pi}^{\pi} f^2\,dx}_{\geq 0 \text{ this integral}} \geq 2a_0^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2).$$

if this integral exists.

In case that f is a square-integrable function ($|f|^2$ is integrable and $\int_{-\pi}^{\pi} |f^2| < +\infty$) on $[-\pi, \pi]$, then Bessel's inequality becomes

<u>Parseval's formula</u>: (= Parseval's identity)

$$\frac{1}{\pi} \int_{-\pi}^{\pi} f^2 dx = 2a_0^2 + \sum_{n=1}^{\infty} (a_n^2 + b_n^2).$$

<u>Example</u>: Minimum square error of the sawtooth wave:

We are looking for the <u>minimum square error</u> E^* of $F(x)$ with $N = 1, 2, \dots, 10, 20, 100, 1000$ relative to

$$f(x) = x + \pi \qquad (-\pi < x < \pi)$$

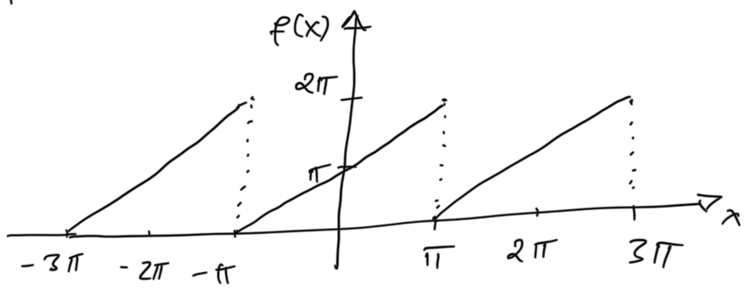
on the interval $[-\pi, \pi]$.

We already know:

$$f(x) = \underbrace{\pi}_{= a_0} + 2\left(\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \dots\right)$$

$$b_1 = 2 \qquad b_2 = -1 = -\frac{2}{2} \qquad b_3 = \frac{2}{3}$$

for $-\pi \leq x \leq \pi$.



We get from (1):

$$E^* = \int_{-\pi}^{\pi} (x+\pi)^2 dx - \pi\left(2\pi^2 + \underbrace{4}_{= 2^2} \sum_{n=1}^{N} \frac{1}{n^2}\right).$$

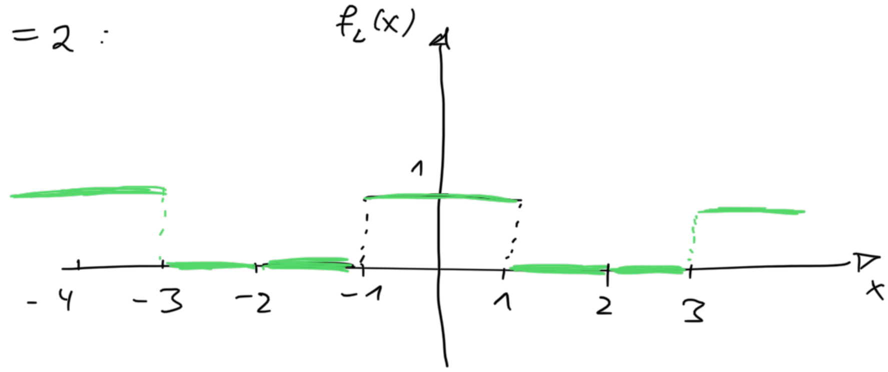
Fourier Integral

- So far, we have been able to represent <u>periodic</u> functions $f_L(x)$ with period $\underline{2L}$.
- <u>Goal</u>: Represent non-periodic functions
- To this end, let us take a periodic function $f_L(x)$ and let L approach infinity.

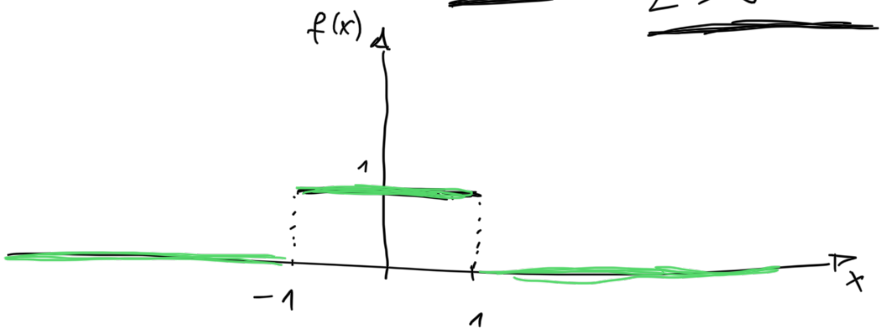
<u>Example</u>: Periodic Rectangular Wave

$$f_L(x) = \begin{cases} 0 & -L < x < -1 \\ 1 & -1 < x < 1 \\ 0 & 1 < x < L \end{cases}.$$

$L = 2$:



If $L \to \infty$, we get $\underline{f(x)} := \underline{\lim_{L \to \infty} f_L(x)} = \begin{cases} 1 & : -1 < x \ldots \\ 0 & : \text{els} \ldots \end{cases}$



- Euler formulas for the Fourier series of $f_L(x)$:
  - Since $f_L(x)$ is even, we have $b_n = \underline{0}$ for

all $n = 1, 2, \dots$

- $\underline{a_0} = \frac{1}{2L} \int_{-1}^{1} 1 \, dx = \underline{\frac{1}{L}}$

- $\underline{a_n} = \frac{1}{L} \cdot \int_{-1}^{1} 1 \cdot \cos \frac{n\pi x}{L} \, dx$

$$= \frac{1}{\cancel{L}} \cdot \frac{\cancel{L}}{n\pi} \cdot \left[\sin \frac{n\pi x}{L} \right]_{-1}^{1}$$

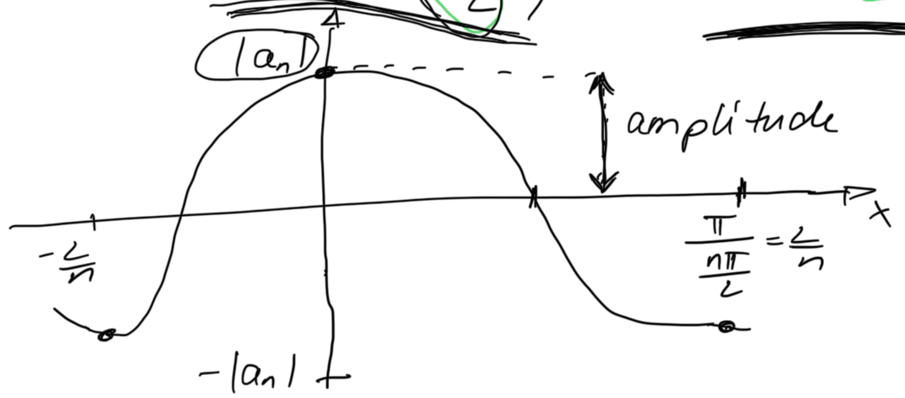
$$\overset{\downarrow}{=} \frac{1}{n\pi} \left(\sin \frac{n\pi}{L} - \underbrace{\sin\left(-\frac{n\pi}{L}\right)}_{= -\sin \frac{n\pi}{L}} \right)$$

$$= \underline{\frac{1}{n\pi} \left(2 \sin \frac{n\pi}{L} \right)}$$

- the <u>sequence</u> of Fourier coefficient $\underline{(a_n)_{n=0}^{\infty}}$ is called the <u>amplitude spectrum</u> of f_L.

(as $|a_n|$ is the maximum amplitude of the wave $\underline{a_n \cdot \cos\left(\frac{n\pi x}{L}\right)}$:

$\underline{\omega_n = \frac{n\pi}{L}}$



- Plot the amplitude spectrum :

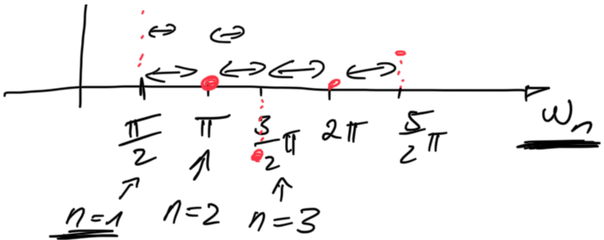
$L = 2$

$\underline{n = 1}$: $\underline{\omega_n} = \frac{\pi}{L} = \underline{\frac{\pi}{2}}$

$\underline{n = 2}$: $\omega_n = \frac{2\pi}{2} = \underline{\pi}$

$n=3: \omega_n = \frac{3\pi}{2}$

$n=4: \omega_n = \frac{4\pi}{2} = \ldots$



- If L increases, the amplitudes get denser.

From Fourier series to Fourier integral

- $f_L(x)$: periodic function with period $2L$
- We assume that $f_L(x)$ can be represented by a Fourier series, i.e.,

$$\boxed{f_L(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos \underbrace{\frac{n\pi}{L}}_{=:\omega_n} x + b_n \sin \underbrace{\frac{n\pi}{L}}_{=\omega_n} x\right)}$$

- Inserting the Euler formulas for a_0, a_n and b_n, we get:

$$f_L(x) = \frac{1}{2L}\int_{-L}^{L} f_L(v)\,dv + \boxed{\frac{1}{L}}\sum_{n=1}^{\infty}\Bigg(\cos(\omega_n x)\underbrace{\int_{-L}^{L} f_L(v)\cdot\cos(\omega_n v)\,dv}_{=a_n\cdot L}$$

$$+ \sin(\omega_n x)\underbrace{\int_{-L}^{L} f_L(v)\cdot \sin(\omega_n v)\,dv}_{=b_n\cdot L}\Bigg)$$

Idea: Let us set $\boxed{\Delta\omega = \omega_{n+1} - \omega_n} = \frac{(n+1)\pi}{L} - \frac{n\pi}{L} = \frac{\pi}{L} = \Delta\omega$

$\omega_{n+1} = \frac{(n+1)\pi}{L}$, $\omega_n = \frac{n\pi}{L}$

$\Longrightarrow \frac{1}{L} = \frac{\Delta\omega}{\pi}$.

We get:

$$f_L(x) = \frac{1}{2L}\int_{-L}^{L} f_L(v)\,dv + \frac{1}{\pi}\sum_{n=1}^{\infty}\Big(\cos\omega_n x\cdot \Delta\omega \cdot \int_{-L}^{L} f_L(v)\cdot$$

$$\cdot \cos\omega_n v\,dv + (\sin\omega_n x)\cdot\Delta\omega \int f_L(v)\cdot \sin\omega_n v \ldots$$

$-L$ dv

let now $L \to \infty$ and we assume that

$$f(x) := \lim_{L\to\infty} f_L(x)$$

is absolutely integrable on the x-axis, this means that

$$\lim_{a\to\infty} \int_{-a}^{0} |f(x)|\,dx + \lim_{b\to\infty} \int_{0}^{b} |f(x)|\,dx$$

exist.

Then we obtain:

$$\lim_{L\to\infty} f_L(x) = 0 + \frac{1}{\pi}\int_0^\infty \Big(\cos wx \cdot \int_{-\infty}^{\infty} f(v)\cdot \cos wv$$
$$+ \sin wx \cdot \int_{-\infty}^{\infty} f(v) \sin wv\, dv \Big) dw$$

Define

$$A(w) := \frac{1}{\pi}\cdot \int_{-\infty}^{\infty} f(v) \cos wv\, dv \qquad a_n$$

$$B(w) := \frac{1}{\pi}\cdot \int_{-\infty}^{\infty} f(v) \sin wv\, dv. \qquad b_n$$

Then

$$\underbrace{\lim_{L\to\infty} f_L(x) = f(x) = \int_0^\infty \big(A(w)\cdot \cos wx + B(w)\cdot \sin wx\big)\, dw}_{\text{representation of } f(x) \text{ by a Fourier integral}}$$