Lecture 18

- Multiplication of (4) by $\cos mx$ (m fixed) and integration yields:

$$\int_{-\pi}^{\pi} f(x)\cos mx\,dx = \int_{-\pi}^{\pi}\Big(a_0 + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx\Big)\cdot \cos mx\,dx$$

$$= a_0 \underbrace{\int_{-\pi}^{\pi}\cos mx\,dx}_{=0} + \sum_{n=1}^{\infty} a_n \Big(\underbrace{\int_{-\pi}^{\pi}\cos nx \cos mx\,dx}_{=\begin{cases}0 & \text{for } n\neq \\ \pi & \text{for } n= \end{cases} \text{ due to (3a}}$$

$$+ b_n \underbrace{\int_{-\pi}^{\pi}\sin nx \cos mx\,dx}_{=0\ \forall n,m \text{ due to (3c)}}\Big)$$

$$= a_m \cdot \pi$$

$$\Longrightarrow \quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

- Multiplying (4) by $\sin mx$ (for fixed m) and integrating yields:

$$\int_{-\pi}^{\pi} f(x)\cdot \sin mx\,dx = \int_{-\pi}^{\pi}\Big(a_0 + \sum_{n=1}^{\infty}(a_n\cos nx + b_n \sin$$

$$\cdot \sin mx\,dx$$

$$= a_0 \underbrace{\int_{-\pi}^{\pi}\sin mx\,dx}_{=0} + \sum_{n=1}^{\infty}\Big(\underbrace{\int_{-\pi}^{\pi} a_n \cos nx \sin mx\,dx}_{=0 \text{ by (3c)}}$$

$$+ \underbrace{\int_{-\pi}^{\pi} b_n \sin nx \sin mx\,dx}\Big)$$

due to (3

$$= \begin{cases} 0 & \text{for } m \neq n \\ b_m \cdot \pi & \text{for } m = n \end{cases} \quad \text{due to } (**)$$

$$= b_m \cdot \pi$$

$$\Longrightarrow \underline{\underline{b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cdot \sin nx \, dx}} .$$

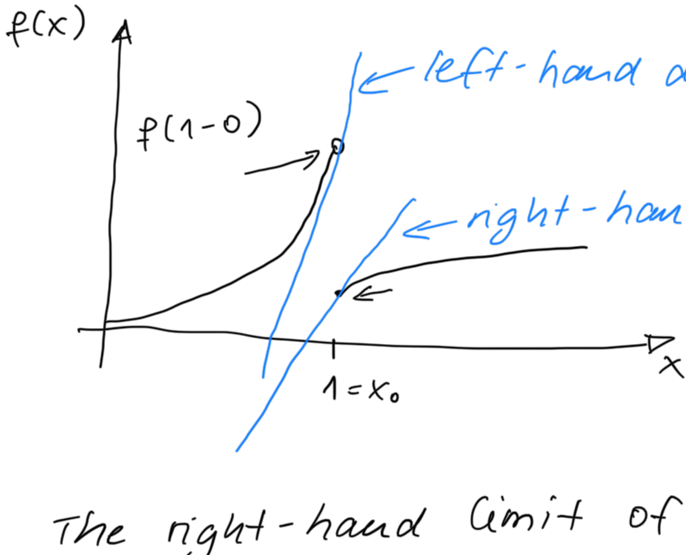
Convergence and Sum of a Fourier Series

Short Repetition:

The left-hand limit of $f(x)$ at x_0 is defined as the limit of $f(x)$ as x approaches x_0 from the left ("$f(x_0 - 0)$")

$$f(x_0 - 0) := \lim_{h \to 0} f(x_0 - h)$$

as $h \to 0$ through positive values.



The right-hand limit of $f(x)$ at x_0 is

$$f(x_0 + 0) := \lim_{h \to 0} f(x_0 + h)$$

as $h \to 0$ through positive values.

The left- and right hand derivatives of $f(x)$ at x_0 are

$$\lim_{\substack{h \to 0 \\ \text{through} \\ \text{positive} \\ \text{values}}} \frac{f(x_0 - h) - f(x_0 - 0)}{-h}$$

and

$$\lim_{\substack{h \to 0 \\ \text{trough} \\ \text{positive} \\ \text{values}}} \frac{f(x_0 + h) - f(x_0 + 0)}{h} .$$

· If f is continuous, then $f(x_0 - 0) = f(x_0 + 0)$
$= f(x_0)$

Theorem (Representation by a Fourier series)
Let $f(x)$ be periodic with period 2π and piecewise continuous in the interval $[-\pi, \pi]$. Let $f(x)$ have a left-hand and right-hand derivative at each point of that interval. Then the Fourier series of f with Fourier coefficients given by the Euler formulas converges. Its sum is $f(x)$, except at points x_0 where $f(x)$ is discontinuous. There, the sum of the series is the average of the left- and right-hand limits of $f(x)$ at x_0.

Proof: We only prove convergence for a continuous function $f(x)$ having continuous first and

function $f(x)$ [illegible]
second derivatives.

We have

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx$$

$$\overset{\text{Integration by parts}}{=} \underbrace{\frac{1}{\pi} \left[\frac{f(x) \sin nx}{n} \right]_{-\pi}^{\pi}}_{=0} - \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{f'(x) \cdot \sin nx}{n} \, dx$$

$$= -\frac{1}{\pi n} \int_{-\pi}^{\pi} f'(x) \sin nx \, dx$$

Int. by parts

$$= +\underbrace{\frac{1}{\pi n} \left[\frac{f'(x) \cos nx}{n} \right]_{-\pi}^{\pi}}_{=0} - \frac{1}{\pi n} \int_{-\pi}^{\pi} \frac{f''(x) \cos nx}{n} \, dx$$

Since f'' is continous, there is M s.t.

$$\boxed{|f''(x)| < M.}$$

Moreover, $\boxed{|\cos nx| \leq 1}$. Therefore,

$$|a_n| = \frac{1}{\pi n^2} \left| \int_{-\pi}^{\pi} f''(x) \cos nx \, dx \right|$$

$$< \frac{1}{\pi n^2} \cdot \int_{-\pi}^{\pi} M \, dx = \frac{2\pi M}{n^2 \pi} = \frac{2M}{n^2}$$

Similarly, $|b_n| < \frac{2M}{n^2}$ for all n.

Hence,

$$\left| a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \right|$$

$$\leq |a_0| + \sum_{n=1}^{\infty} \left(|a_n| \cdot |\cos nx| + |b_n| \cdot |\sin nx| \right)$$

$$\leq |a_0| + \sum_{n=1}^{\infty} \left(\frac{2M}{n^2} + \frac{2M}{n^2} \right)$$

$$= |a_0| + \sum_{n=1}^{\infty} \frac{4M}{n^2} = |a_0| + 4M \cdot \overbrace{\underbrace{\sum_{n=1}^{\infty} \frac{1}{n^2}}_{= \frac{\pi^2}{6}}}^{\text{converges!}} .$$

$\Rightarrow$ The Fourier series converges. ■

Arbitrary Period: From Period 2π to any Period $p = 2L$

Goal: Represent functions of any period p ($p = 2L$) by a Fourier series.

$\Rightarrow$ Change of scale: let $f(x)$ have the period $p = 2L$. Set $x = \frac{p}{2\pi} \cdot v$

new variable

$$\Rightarrow v = \frac{2\pi x}{p} = \frac{\pi x}{L} \quad \Rightarrow x = \frac{Lv}{\pi}.$$

$\Rightarrow$ If $v = \pm\pi$, then $x = \pm L$ and vice versa

$$\Rightarrow f(x) = f\left(\frac{Lv}{\pi}\right) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nv + b_n \sin nv$$

with coefficients:

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f\left(\frac{Lv}{\pi}\right) dv,$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f\left(\frac{Lv}{\pi}\right) \cos nv \, \underline{dv},$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f\left(\frac{Lv}{\pi}\right) \cdot \sin nv \, \underline{dv}.$$

Since $v = \frac{\pi x}{L}$, we have $\frac{dv}{dx} = \frac{\pi}{L}$

$$\Rightarrow \underline{\underline{dv = \frac{\pi}{L} \cdot dx}}$$

Integration boundaries:

$$v = -\pi \quad \Rightarrow \quad x = \frac{L \cdot (-\pi)}{\pi} = \underline{\underline{-L}}$$

$$v = \pi \quad \Rightarrow \quad x = \frac{L \cdot \pi}{\pi} = \underline{\underline{L}}$$

We obtain for a function $f(x)$ of period $2L$ the Fourier series:

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi}{L} x\right) + b_n \sin\left(\frac{n\pi}{L} x\right)\right)$$

with Fourier coefficients

$$\underline{\underline{a_0}} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f\left(\frac{Lv}{\pi}\right) dv$$

$$= \frac{1}{2\cancel{\pi}} \int_{-L}^{L} f\Big(\underbrace{\frac{Lv}{\pi}}_{=x}\Big) \cdot \frac{\cancel{\pi}}{L} dx$$

$$= \underline{\underline{\frac{1}{2L} \int_{-L}^{L} f(x) \, dx}}$$

$$a_n = \frac{1}{\ } \int^{\pi} f\left(\frac{Lv}{\ }\right) \cos nv \, dv$$

$$= \frac{1}{\cancel{\pi}} \int_{-L}^{L} f(x) \cdot \cos\left(\frac{n\pi x}{L}\right) \cdot \frac{\cancel{\pi}}{L}\, dx$$

$$= \frac{1}{L} \int_{-L}^{L} f(x) \cdot \cos\left(\frac{n\pi x}{L}\right) dx$$

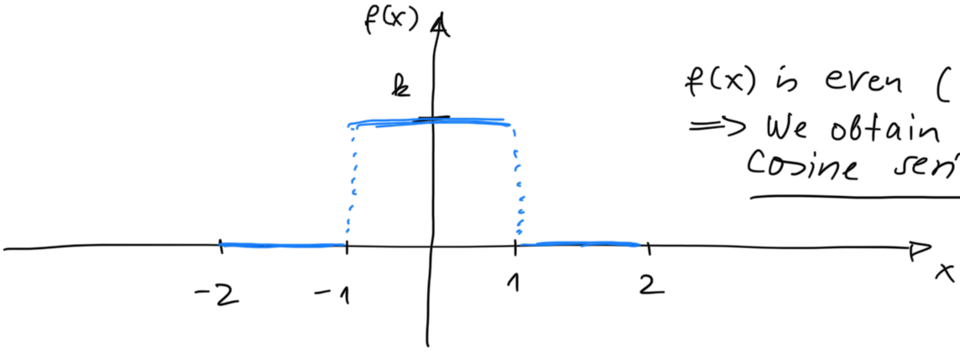
$$\underline{\underline{b_n}} = \frac{1}{\pi} \int_{-\pi}^{\pi} f\left(\frac{Lv}{\pi}\right) \sin nv\, dv$$

$$= \frac{1}{\cancel{\pi}} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) \frac{\cancel{\pi}}{L}\, dx$$

$$= \frac{1}{L} \int_{-L}^{L} f(x) \cdot \sin\left(\frac{n\pi x}{L}\right) dx$$

<u>Example</u>: Periodic Rectangular Wave

$$f(x) = \begin{cases} 0 & \text{if } -2 < x < -1 \\ k & \text{if } -1 < x < 1 \\ 0 & \text{if } 1 < x < 2. \end{cases}$$



$f(x)$ is even ($f(x) = f(-$
$\Rightarrow$ We obtain a <u>Fouri</u>
<u>Cosine series</u>

$$p = 2L = 4 \quad (L = 2)$$

We get:

$$a_0 = \frac{1}{\;} \int^{L} f(x)\, dx = \frac{1}{\;} \int^{2} f(x)\, dx$$

$2L \quad -L \qquad\qquad 4 \quad -2 \quad = k$ for $x \in [-1,1]$

$$= \frac{1}{4} \cdot k \left[x \right]_{-1}^{1} = \frac{1}{4} k \left(1 - (-1) \right) = \underline{\underline{\frac{k}{2}}}$$

$$\underline{\underline{a_n}} = \frac{1}{L} \int_{-L}^{L} f(x) \cdot \cos\left(\frac{n\pi x}{L}\right) dx$$

$$= \frac{1}{2} \int_{-1}^{1} k \cdot \cos\left(\frac{n\pi x}{2}\right) dx = \frac{k}{\cancel{2}} \left[\frac{\cancel{2}}{n\pi} \sin \frac{n\pi x}{2} \right]_{-1}^{1}$$

$$= \frac{k}{n\pi} \left(\sin\left(\frac{n\pi}{2}\right) - \underbrace{\sin\left(-\frac{n\pi}{2}\right)}_{= - \sin\left(\frac{n\pi}{2}\right)} \right)$$

$$= \underline{\underline{\frac{2k}{n\pi} \sin\left(\frac{n\pi}{2}\right)}}$$

$$\underline{\underline{b_n}} = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx = \frac{1}{2} \int_{-1}^{1} k \sin\left(\frac{n\pi x}{2}\right) d$$

$$= \frac{k}{2} \left[- \frac{2}{n\pi} \cos\left(\frac{n\pi x}{2} \right) \right]_{-1}^{1}$$

$$= \frac{k}{2} \left(- \frac{2}{n\pi} \underbrace{\cos \frac{n\pi}{2}}_{=0} - \left(- \frac{2}{n\pi} \underbrace{\cos\left(-\frac{\pi n}{2}\right)}_{=0} \right) \right)$$

$$= \underline{\underline{0}}$$

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) \qquad = \begin{cases} \pm 1 & : n \text{ odd} \\ 0 & : n \text{ even} \end{cases}$$

$$= \frac{k}{2} + \frac{2k}{\pi} \cdot \sum_{n=1}^{\infty} \frac{1}{n} \cdot \overbrace{\sin\left(\frac{n\pi}{2}\right)} \cdot \cos\left(\frac{n\pi x}{L}\right)$$

$$= \frac{k}{2} \cdot \frac{2k}{\pi}\Big(\underbrace{\cos\left(\frac{\pi}{2}x\right)}_{n=1} - \underbrace{\frac{1}{3}\cdot\cos\left(\frac{3\pi x}{2}\right)}_{n=3} + \underbrace{\frac{1}{5}\cos\left(\frac{5\pi x}{2}\right)}_{n=5} - \dots \Big)$$

Example: Let $f(x) = \begin{cases} -k & : \ -2 < x < 0 \\ k & : \ 0 < x < 2 \end{cases}$

$\Rightarrow p = 2L = 4 \quad \Rightarrow L = 2.$

This is the "basic example" from Lecture 17, now with $p = 4$ instead of 2π.

Since $v = \frac{\pi}{L} \cdot x = \frac{\pi}{2} \cdot x$, and we know the Fourier series for $p = 2\pi$:

$$\frac{4k}{\pi}\left(\sin v + \frac{1}{3}\sin 3v + \frac{1}{5}\sin 5v + \dots\right.$$

We get:

$$\underline{\underline{f(x) = \frac{4k}{\pi}\cdot\left(\sin\left(\frac{\pi}{2}x\right) + \frac{1}{3}\sin\left(\frac{3\pi}{2}x\right) + \frac{1}{5}\sin\left(\frac{5\pi}{2}x\right) + \dots\right)}}.$$

Simplifications: Even and Odd functions

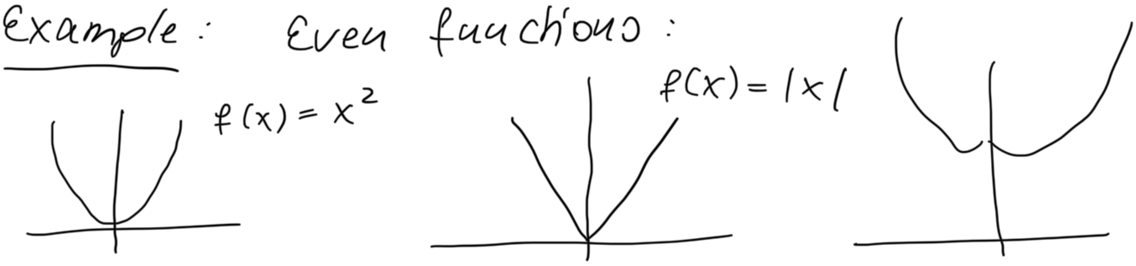
Definition: $f(x)$ is an <u>even</u> function if

$$\boxed{f(x) = f(-x)}$$

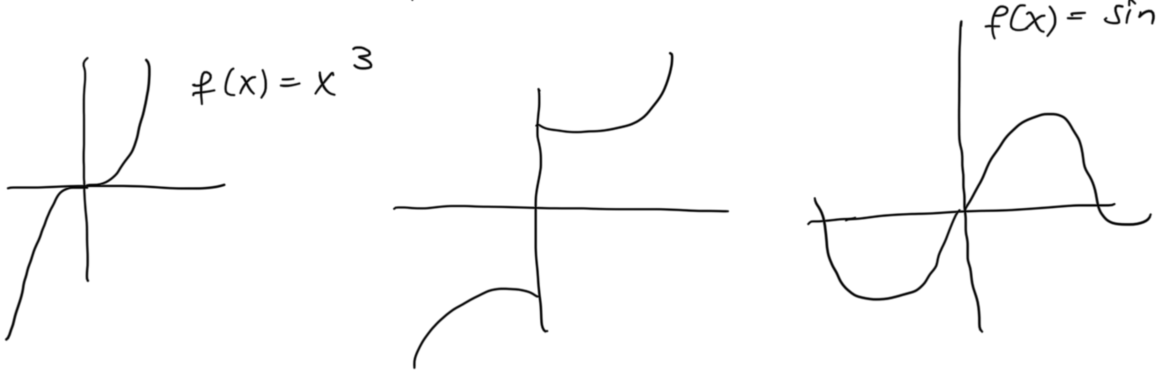
$f(x)$ is an <u>odd</u> function if

$$\boxed{f(-x) = -f(x)}$$

<u>Example</u>: Even functions:



Odd functions:



- If $f(x)$ is an <u>even</u> function, its Fourier seri reduces to a <u>Fourier cosine series</u>, that is

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi}{L} x$$

with coefficients

$$\underbrace{a_0 = \frac{1}{L} \int_0^L f(x)\,dx}_{= \frac{1}{2L} \int_{-L}^{L} f(x)\,dx}$$

L

$$a_n = \frac{2}{L} \int_0^L \underbrace{\underbrace{f(x)}_{\text{even}} \cdot \underbrace{\cos\left(\frac{n\pi x}{L}\right)}_{\text{even}}}_{\text{even}} \, dx \quad , \quad n = 1, 2, 3, \ldots$$

Note : $$b_n = \frac{1}{L} \int_{-L}^{L} \underbrace{\underbrace{f(x)}_{\text{even}} \cdot \underbrace{\sin\left(\frac{n\pi x}{L}\right)}_{\text{odd}}}_{\text{odd}} \, dx = 0$$