NTNU | Kunnskap for en bedre verden

TMA4130 MATEMATIKK 4N

Lecture 9: Laplace Transform. Linearity. First Shifting Theorem (s-Shifting)

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Important!

Remember to use **check-in** for this room!

Plan for the day

After today's lecture, you should be familiar with

- use and definition of the Laplace transform
- how to compute the Laplace transform
- first shifting theorem (s-shifting)
- existence of Laplace transforms.

Solving linear ODEs and related initial value problems

Initial Value Problem (IVP) ODE $\xrightarrow[\text{Transform}]{\text{Laplace}}$ Algebraic Problem (AP) $\Longrightarrow$ Solving AP by Algebra $\Longrightarrow$ Solution of the IVP

$t \ldots$ time

Initial Value Problem
$$\begin{cases} y''(t) - 2y(t) = -e^{t} & \text{ODE} \\ \left.\begin{aligned} y(0) &= 1 \\ y'(0) &= 1 \end{aligned}\right\} & \text{initial values} \end{cases}$$

What is a transform?

Input : function
Output : function

The idea of a transform is that it turns a given function into another function that may not be in the same domain. We are already acquainted with several transforms:

1. The derivative D takes a differentiable function f (defined on some interval (a, b)) and assigns to it a new function $\boxed{Df := f'}$.
2. The integral I takes a continuous function f (defined on some interval $[a, b]$) and assigns to it a new function
$$\boxed{If(t) := \int_a^t f(x)dx.}$$
3. The multiplication operator M_ϕ, which multiplies any given function f on the interval $[a, b]$ by a fixed function ϕ on $[a, b]$, is a transform:
$$\boxed{M_\phi f(t) := \phi(t) \cdot f(t).}$$

We are particularly interested in transforms that are linear. A transform T is *linear* if for some functions f, g and constants $\alpha, \beta \in \mathbb{R}$

$$\boxed{T[\alpha f + \beta g] = \alpha T(f) + \beta T(g).}$$

In particular (taking $\alpha = \beta = 1$),

$$\boxed{T[f + g] = T(f) + T(g),}$$

and (taking $\beta = 0$)

$$\boxed{T[\alpha f] = \alpha T(f).}$$

Definition: Laplace transform

$= \{t \in \mathbb{R} \mid t \geq 0\}$

Given a function $f(t)$ (t = time, $f : \mathbb{R}_+ \to \mathbb{R}$), its **Laplace transform** is defined as

Laplace

$$F(s) := \mathcal{L}(f) := \int_0^\infty e^{-st} f(t)dt.$$

The given function $f(t)$ is called the **inverse transform** of $F(s)$ and is denoted by $\mathcal{L}^{-1}(F)$. Note that the above integral is an **improper integral**, which is evaluated according to the rule

$$\int_0^\infty e^{-st} f(t)dt := \lim_{T\to\infty} \int_0^T e^{-st} f(t)dt.$$

Laplace Transform

$$\boxed{F(s) := \mathcal{L}(f) := \int_0^{\infty} e^{-st} f(t)\,dt}$$

Remark 1
The Laplace transform is only concerned with $f(t)$ for $t \geq 0$. Thus, we can set $f(t) = 0$ for $t < 0$.

Remark 2
Original functions are denoted by lowercase letters and their transforms by the same letters in capital, so that $F(s)$ denotes the transform of $f(t)$, and $Y(s)$ denotes the transform of $y(t)$, and so on.

Laplace Transform

- named after Pierre-Simon, marquis de Laplace, who used a similar transform on his additions to the probability theory

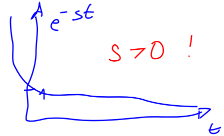
Figure: Pierre-Simon, marquis de Laplace, 23 March 1749 – 5 March 1827, French astronomer and mathematician.

$$\boxed{F(s) := \mathcal{L}(f) := \int_0^\infty e^{-st} f(t)\,dt}$$

Example

Let $f(t) = 1$ when $t \geqq 0$. Find $F(s)$.

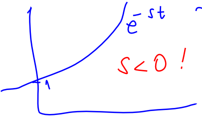
$$\mathcal{L}(f) = \int_0^\infty e^{-st}\cdot 1\,dt = \lim_{T\to\infty}\int_0^T e^{-st}\cdot 1\,dt = \lim_{T\to\infty}\left[-\frac{1}{s}e^{-st}\right]_0^T$$



$$= \underbrace{\lim_{T\to\infty} -\frac{1}{s}e^{-sT}}_{=0\ (s>0)} - \underbrace{\left(-\frac{1}{s}\cdot\underbrace{e^{-s0}}_{=1}\right)}_{}$$

$$= 0 + \frac{1}{s}$$

$$= \frac{1}{s} \quad (\text{if } s>0)$$



$$\int_0^\infty e^{-st}\cdot 1\,dt = +\infty$$

$\Rightarrow \mathcal{L}(f)$ does not exist.

$s=0$: $\int_0^\infty e^{0}\cdot 1 = \int_0^\infty 1\,dt = +\infty \Rightarrow \mathcal{L}(f)$ does not exist.

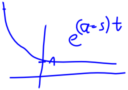
$$F(s) := \mathcal{L}(f) := \int_0^\infty e^{-st} f(t)\,dt$$

Example

Let $f(t) = e^{at}$ when $t \geq 0$, where a is a constant. Find $F(s)$.

$$\mathcal{L}(f) = \int_0^\infty e^{-st} \cdot e^{at}\,dt = \int_0^\infty e^{(a-s)t}\,dt = \lim_{T\to\infty}\left[\frac{1}{a-s}\cdot e^{(a-s)t}\right]_0^T$$

• $a-s>0 \Rightarrow \mathcal{L}(f)$ does not exist.



• $a-s<0 \Rightarrow \mathcal{L}(f) = \underbrace{\lim_{T\to\infty} \frac{1}{a-s} e^{(a-s)T}}_{=0} - \frac{1}{a-s}\cdot \underbrace{e^0}_{=1} = \frac{1}{s-a}$

• $a=s \Rightarrow \mathcal{L}(f) = \int_0^\infty e^{0\cdot t}\,dt = \int_0^\infty 1\,dt = \infty \Rightarrow \mathcal{L}(f)$ does not exist.

If $a=0 \Rightarrow f(t) = e^0 = 1 \Rightarrow \mathcal{L}(f) = \frac{1}{s}$ $(s>0)$ in accordance with the preceding example.

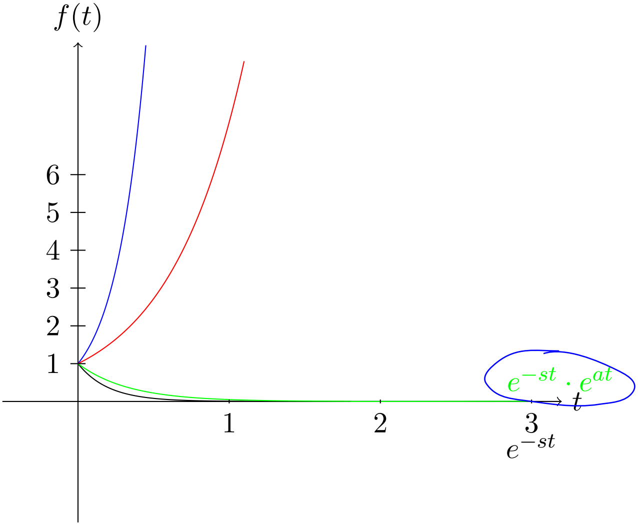


Figure: Visual verification that the Laplace transform of e^{at} exists only if $s > a$: Functions e^{st} with $s = 5$ and $f(t) = e^{at}$ with $a = 2$.

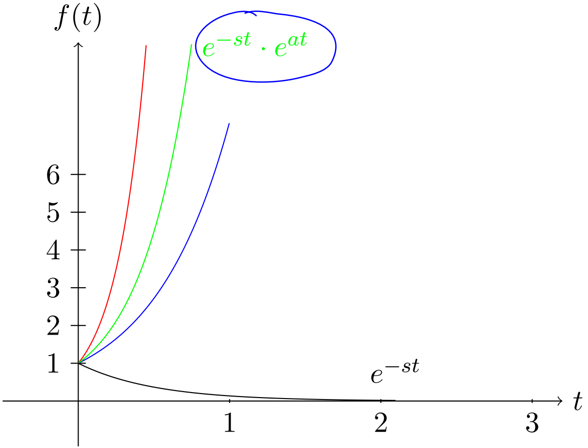


Figure: Visual verification that the Laplace transform of e^{at} does not exists if $s < a$: Functions e^{st} with $s = 2$ and $f(t) = e^{at}$ with $a = 5$.

A Non-Working Example

$$F(s) := \mathcal{L}(f) := \int_0^\infty e^{-st} f(t)dt$$

Example

Does the Laplace transform of $f(t) = e^{t^2}$ exist?

$$\mathcal{L}(f) = \int_0^\infty e^{-st} \cdot e^{t^2} dt$$

$$= \int_0^\infty e^{t(t-s)} dt \qquad (t \to \infty \text{, at one point, } t \text{ will be bigger than } s)$$

$$= \infty$$

$\Rightarrow e^{t^2}$ increase much faster than e^{-st} decreases.

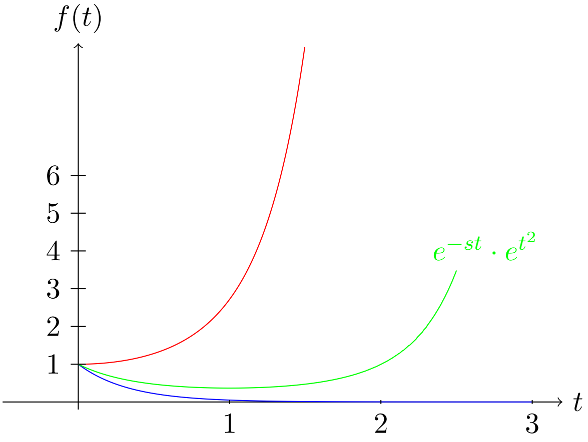


Figure: Functions $f(t) = e^{t^2}$ and e^{-st} for $s = 2$.

$\mathcal{L}(f)$, $\mathcal{L}\{a \cdot f\}$

Theorem: Linearity of the Laplace Transform

The Laplace transform is a linear operation; that is, for any functions f and g whose transforms exist and any constants a and b the transform of $af(t) + bg(t)$ exists, and

$$\boxed{\mathcal{L}\{af(t) + bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}.}$$

Proof: $\mathcal{L}\{af(t) + bg(t)\} = \int_0^\infty e^{-st}(af(t) + bg(t))\,dt$

$= a\int_0^\infty e^{-st} f(t)\,dt + b\int_0^\infty e^{-st} g(t)\,dt$

$= a \cdot \mathcal{L}(f) + b \cdot \mathcal{L}(g)$.

Example

Find the transforms of $\cosh at$ and $\sinh at$.

$$\boxed{\cosh at = \frac{1}{2}\left(e^{at} + e^{-at}\right)}$$

$$\mathcal{L}(\cosh at) = \frac{1}{2}\underbrace{\mathcal{L}(e^{at})} + \frac{1}{2}\underbrace{\mathcal{L}(e^{-at})}$$

$$= \frac{1}{2}\cdot\frac{1}{s-a} + \frac{1}{2}\cdot\frac{1}{s+a}$$

($s > a$ and $s > -a$
$\Rightarrow s > |a|$)

$$= \frac{1}{2}\left(\frac{s+\cancel{a} + s-\cancel{a}}{(s-a)(s+a)}\right)$$

$$= \frac{1}{\cancel{2}}\cdot\frac{\cancel{2}s}{s^2+\cancel{sa}-\cancel{sa}-a^2}$$

$$= \frac{s}{s^2-a^2}$$

$$\sinh at = \frac{1}{2}\left(e^{at} - e^{-at}\right)$$

$$\mathcal{L}(\sinh at) = \frac{1}{2} \cdot \mathcal{L}\left(e^{at}\right) - \frac{1}{2}\mathcal{L}\left(e^{-at}\right)$$

$$= \ldots\ldots = \underline{\underline{\frac{a}{s^2 - a^2}}}$$

Example

Derive the formulas

$$\mathcal{L}(\cos \omega t) = \frac{s}{s^2 + \omega^2}$$

and

$$\mathcal{L}(\sin \omega t) = \frac{w}{s^2 + \omega^2}.$$

Integration by parts:

$$\int g' f = g \cdot f - \int g f'$$

$$\mathcal{L}(\cos \omega t) = \int_0^\infty \underbrace{e^{-st}}_{g'} \cdot \underbrace{\cos \omega t}_{f} \, dt$$

$$= \Big[\underbrace{-\frac{1}{s} e^{-st}}_{g} \cdot \cos \omega t \Big]_0^{\infty} - \int_0^\infty -\frac{1}{s} e^{-st} \cdot (-\omega \sin \omega t) \, dt$$

$$= 0 - \left(-\frac{1}{s} \cdot 1 \cdot 1 \right) - \frac{\omega}{s} \underbrace{\int_0^\infty e^{-st} \sin \omega t \, dt}_{= \mathcal{L}(\sin \omega t)}$$

$$= \frac{1}{s} - \frac{\omega}{s} \mathcal{L}(\sin \omega t)$$

$$\underbrace{\mathcal{L}(\sin \omega t)}_{=:\mathcal{L}_s} = \int_0^\infty \underbrace{e^{-st}}_{g'} \underbrace{\sin \omega t}_{f} \, dt$$

$$= \left[-\frac{1}{s} e^{-st} \sin \omega t \right]_0^\infty + \int_0^\infty \frac{\omega}{s} e^{-st} \cos \omega t \, dt$$

$$= 0 - \left(-\frac{1}{s} \cdot 1 \cdot 0 \right) + \frac{\omega}{s} \cdot \mathcal{L}(\cos \omega t)$$

$$= \frac{\omega}{s} \cdot \underbrace{\mathcal{L}(\cos \omega t)}_{=:\mathcal{L}_c}$$

$$\left. \begin{aligned} \mathcal{L}_c &= \frac{1}{s} - \frac{\omega}{s} \cdot \mathcal{L}_s \\ \mathcal{L}_s &= \frac{\omega}{s} \mathcal{L}_c \end{aligned} \right\} \quad \mathcal{L}_c = \frac{1}{s} - \frac{\omega^2}{s^2} \mathcal{L}_c \Rightarrow \frac{1}{s} = \mathcal{L}_c + \frac{\omega^2}{s^2} \mathcal{L}_c$$

$$= \frac{s^2 + \omega^2}{s^2} \cdot \mathcal{L}_c$$

$$\Rightarrow \mathcal{L}_c = \frac{1}{\cancel{s}} \cdot \frac{s^{\cancel{2}}}{s^2 + \omega^2} = \underline{\underline{\frac{s}{s^2 + \omega^2}}}$$

$$\Rightarrow \mathcal{L}_s = \frac{\omega}{s} \cdot \mathcal{L}_c = \frac{\omega}{\cancel{s}} \cdot \frac{\cancel{s}}{s^2 + \omega^2} = \underline{\underline{\frac{\omega}{s^2 + \omega^2}}}$$

Some Functions $f(t)$ and Their Laplace Transforms $\mathcal{L}(f)$

	$f(t)$	$\mathcal{L}(f)$
1	1	$1/s$
2	t	$1/s^2$
3	t^2	$2!/s^3$
4	t^n $(n = 0, 1, \cdots)$ t^3	$\dfrac{n!}{s^{n+1}}$ $\dfrac{3!}{s^4}$
5	t^a (a positive)	$\dfrac{\Gamma(a+1)}{s^{a+1}}$
6	e^{at}	$\dfrac{1}{s-a}$

	$f(t)$	$\mathcal{L}(f)$
7	e^{at} $\cos \omega t$	$\boxed{\dfrac{s}{s^2+\omega^2}}$ $F(s-a)$
8	$\sin \omega t$	$\dfrac{\omega}{s^2+\omega^2}$
9	$\cosh at$	$\dfrac{s}{s^2-a^2}$
10	$\sinh at$	$\dfrac{a}{s^2-a^2}$
11	$e^{at} \cos \omega t$	$\dfrac{s-a}{(s-a)^2+\omega^2}$
12	$e^{at} \sin \omega t$	$\dfrac{\omega}{(s-a)^2+\omega^2}$

Example

Find the Laplace transform of the function

$$f(t) = 5t^3 - 2e^t.$$

$$\mathcal{L}(f) = 5 \cdot \mathcal{L}(t^3) - 2 \cdot \mathcal{L}(e^t) \qquad (e^{at},\ a = 1)$$

$$= 5 \cdot \frac{3!}{s^4} - 2 \cdot \frac{1}{s-1}$$

Recap

$$\boxed{F(s) := \mathcal{L}(f) := \int_0^\infty e^{-st} f(t)dt}$$

What we have learned so far

- Introduction of Laplace transform; computing of the Laplace transform; a first property (linearity)
- Laplace transform is a technique for solving differential equations
- differential equation of time domain form is first transformed to algebraic equation of frequency domain form. After solving the algebraic equation in frequency domain, the result then is finally transformed to time domain form to give the solution of the differential equation

What's Ahead

What we will cover in the remainder of today's lecture

- A first shifting theorem
- Existence and uniqueness of Laplace transform

Theorem: First Shifting Theorem, s-Shifting

If $f(t)$ has the transform $F(s)$ (where $s > k$ for some k), then $e^{at}f(t)$ has the transform $F(s-a)$ (where $s-a>k$). In formulas,

$$\mathcal{L}\{e^{at}f(t)\} = F(s-a),$$

or, if we take the inverse on both sides,

$$e^{at}f(t) = \mathcal{L}^{-1}\{F(s-a)\}.$$

Proof: $\mathcal{L}\{e^{at}f(t)\} = \int_0^\infty e^{-st} \cdot e^{at} f(t)\,dt = \underbrace{\int_0^\infty e^{-(s-a)t} f(t)\,dt}_{F(s-a)}$

$= F(s-a)$.

If $F(s)$ exists (i.e., is finite) for $s > k$, then $F(s-a)$ exists for $s-a > k$.

Example

Find the inverse of the transform

$$\mathcal{L}(f) = \frac{3s - 137}{s^2 + 2s + 401}.$$

We know: $\mathcal{L}\{e^{at}\cos wt\} = \dfrac{s-a}{(s-a)^2 + w^2}$

$\mathcal{L}\{e^{at}\sin wt\} = \dfrac{w}{(s-a)^2 + w^2}$ $\quad \mathcal{L}^{-1}$ linear

$$f = \mathcal{L}^{-1}\left\{\frac{3(s+1) - 140}{(s+1)^2 + 400}\right\} = 3\cdot\mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2 + 20^2}\right\} - 7\cdot\mathcal{L}^{-1}\left\{\frac{20}{(s+1)^2 + 20^2}\right\}$$

$= -a$, $= w$, w, $= -a$, w

$$= 3\cdot e^{-t}\cdot\cos 20t - 7\cdot e^{-t}\cdot\sin 20t$$

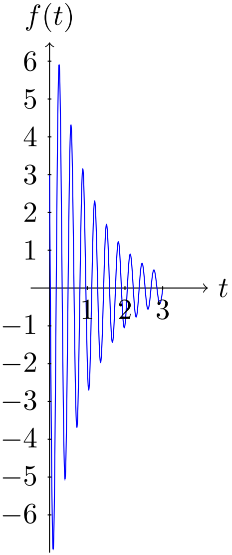


Figure: Function $f(t) = e^{-t} \cdot (3\cos 20t - 7\sin 20t)$.

Existence and Uniqueness of Laplace Transforms

1. $f(t)$ should satisfy the **growth restriction**

$$\boxed{\exists\, M, k \; s.t. \; \forall t \geqq 0 : \; |f(t)| \leqq Me^{kt}.}$$ (2)

2. $f(t)$ should be **piecewise** continuous on a finite interval $a \leqq t \leqq b$ where f is defined. That is, this interval can be divided into finitely many subintervals in each of which f is continuous and has **finite** limits as t approaches either endpoint of such a subinterval from the interior.

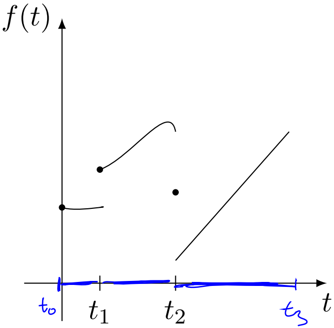


Figure: A piecewise continuous function $f(t)$

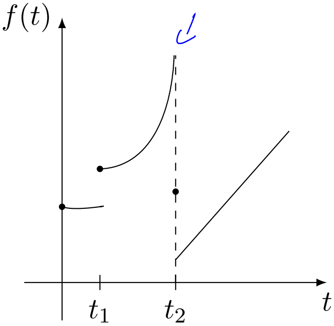


Figure: $f(t)$ is **not** piecewise continuous.

Theorem: Existence Theorem for Laplace Transforms

If $f(t)$ is defined and piecewise continuous on every finite interval on the semi-axis $t \geqq 0$ and satisfies (2) for all $t \geqq 0$ and some constants M and k, then the Laplace transform $\mathcal{L}(f)$ exists for all $s > k$.

Proof: Since $f(t)$ is piecewise continuous, $e^{-st} \cdot f(t)$ is integrable over any finite interval on the t-axis.

$$|\mathcal{L}(f)| = \left| \int_0^\infty e^{-st} f(t)\, dt \right| \leq \int_0^\infty |f(t)| \cdot e^{-st}\, dt \overset{\text{growth restriction (2)}}{\leq} \int_0^\infty M \cdot e^{kt} \cdot e^{-st}\, dt$$

$$= \frac{M}{s-k} \quad (\text{if } s > k)$$

$\Rightarrow \mathcal{L}(f)$ exists for all $s > k$.

Uniqueness

If the Laplace transform of a given function exists, it is uniquely determined. Conversely, it can be shown that if two functions (both defined on the positive real axis) have the same transform, these functions cannot differ over an interval of positive length, although they may differ at isolated points. Hence we may say that the inverse of a given transform is essentially unique. In particular, if two continuous functions have the same transform, they are completely identical.

Conclusion

$$\boxed{F(s) := \mathcal{L}(f) := \int_0^{\infty} e^{-st} f(t)dt}$$

What we have learned today

- Introduction of Laplace transform; computing of the Laplace transform; a first property (linearity)
- Laplace transform is a technique for solving differential equations
- differential equation of time domain form is first transformed to algebraic equation of frequency domain form. After solving the algebraic equation in frequency domain, the result then is finally transformed to time domain form to give the solution of the differential equation
- A first shifting theorem
- Existence and uniqueness of Laplace transform

Next Lecture

Chapter 6.2 in Kreyszig

- Laplace transforms of derivatives and integrals
- ODEs

References

The material of this lecture was based on Chapter 6.1 of the book

Advanced Mathematical Engineering by Erwin Kreyszig (John Wiley & Sons, 10th edition, 2011)

and Chapter 6 in

Differential Equations Demystified by Steven G. Krantz (McGraw-Hill, 2005).

Moreover, we recommend the lecture notes by Morten Nome (in Norwegian), who taught the 2019 edition of this course. You can download Lecture 1 of Morten's lecture notes collection here:

https://www.math.ntnu.no/emner/TMA4125/2019v/notater/01-laplacetransform.pdf