NTNU | Kunnskap for en bedre verden

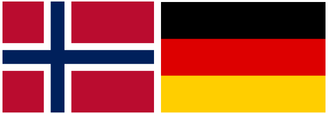
TMA4130 MATEMATIKK 4N

Lecture 7: Numerical Methods for Nonlinear Equations

Elisabeth Köbis

September 13, 2021

Before we start / Før vi kan starte / Bevor es losgeht



About me

- Elisabeth Köbis, elisabeth.kobis@ntnu.no, +47 73412539.
- Room 1054 Sentralbygg II. Office hours: Tuesdays, 12.00-14.00 (Please send me an email to schedule an appointment beforehand.)
- Research: Optimization (of set-valued mapping); vector programming; uncertain/robust optimization; (vector) variational inequalities;...

Important!

Remember to use **check-in** for this room!

Outline for the Day

- Introduction
  - Problem formulation
  - Definition of a root
  - Iterative methods
  - Linear convergence
- Bisection Method
- Fixed Point Iteration

Bisection Method: Example

The function $f : [a, b] \to \mathbb{R}$ is Lipschitz continuous[1] with Lipschitz constant $L > 0$ has exactly one root $r \in [a, b]$.

1. Determine the number k of iteration steps that are necessary to enclose the root r with the bisection method in an interval I_k of length $\delta > 0$.
2. Compute the number k of iterations necessary to obtain an interval I_k, such that for a given $\epsilon > 0$, we have

$$\max_{x \in I_k} |f(x)| < \epsilon.$$

[1] $|f(x_1) - f(x_2)| \leq L \cdot |x_1 - x_2|$ for any $x_1, x_2 \in [a, b]$

Bisection Method: Solution to the Example I

1. We have

$$\underbrace{|I_k|}_{\text{length of the } k\text{th interval}} = |b_k - a_k| = \frac{b-a}{2^k},$$

solving for k gives

$$k \geq \log_2 \left(\frac{b-a}{|I_k|} \right).$$

Therefore, $|I_k| \leq \delta$ for $k \geq \log_2 \frac{b-a}{\delta}$.

Bisection Method: Solution to the Example II

2. Moreover,

$$\max_{x\in I_k}|f(x)| = |\max\{\max f(x), -\min f(x)\}| \leq |\max f(x) + (-\min f(x))|$$

$$= |\max f(x) - \min f(x)| \leq L\,|I_k| = \frac{L(b-a)}{2^k}.$$

The maximum is hence smaller than ϵ if

$$\frac{L(b-a)}{2^k} < \epsilon,$$

i.e., if

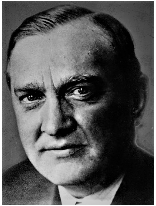
$$k > \log_2\left(\frac{L(b-a)}{\epsilon}\right).$$

□

Fixed Point Iterations

Stefan Banach

Stefan Banach (born in 1892 in Kraków, Austria-Hungary and died 31 August 1945 in Lvov, Ukrainian SSR, Soviet Union) was a Polish mathematician who is generally considered one of the world's most important and influential 20th-century mathematicians. He was one of the founders of modern functional analysis, and an original member of the Lwów School of Mathematics. His major work was the 1932 book, Théorie des opérations linéaires (Theory of Linear Operations), the first monograph on the general theory of functional analysis.



Fixed Point Iterations: Example

Example

The function $g : \mathbb{R} \to \mathbb{R}$ with $g(x) = \sqrt{1 + \frac{x^2}{2}}$ generates the sequence $(x_k)_{n\in\mathbb{N}}$ by $x_{k+1} = g(x_k)$, $k = 0, 1, \ldots$. Show that the sequence (x_k) has a uniquely determined fixed point, and calculate the fixed point.

Fixed Point Iterations: Solution to the Example

The derivative of g reads

$$\frac{dg}{dx} = \frac{1}{2} \cdot \left(1 + \frac{x^2}{2}\right)^{-\frac{1}{2}} \cdot 2x = \frac{x}{\sqrt{\frac{2+x^2}{2}}} = \frac{x}{\sqrt{2} \cdot \sqrt{2 + x^2}}$$

We check the contraction property $|g'(x)| < 1$, i.e.,

$$\frac{|x|}{\sqrt{2} \cdot \sqrt{2 + x^2}} < \frac{1}{\sqrt{2}} < 1, \qquad x \in \mathbb{R},$$

because the denumerator is always bigger than $|x|$. We make the ansatz

$$\tilde{x} = \sqrt{1 + \frac{\tilde{x}^2}{2}}$$

an get $\tilde{x}^2 = 1 + \frac{\tilde{x}^2}{2}$, and eventually $\tilde{x}^2 = 2$. Hence, $\tilde{x} = \sqrt{2}$ (note that the negative solution cancels due to the definition of the sequence).

Resources

- Lecture notes on numerical solutions of nonlinear equations by Morten Nome (in Norwegian): https://www.math.ntnu.no/emner/TMA4125/2019v/notater/06-likningslosere.pdf
- Lecture notes on numerical solutions of nonlinear equations by Anne Kværnø: https://wiki.math.ntnu.no/_media/tma4130/2020h/nonlinearequations.pdf
- Book *Numerical Mathematics and Computing* by Ward Cheney and David Kincaid, 7th edition, 2013, Brooks/Cole.

Next lecture

- Newton's Method
- Systems of Nonlinear Equations