NTNU | Kunnskap for en bedre verden

TMA4130 MATEMATIKK 4N

Lecture 21: Fourier Integral

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Repetition: Fourier series

Consider a periodic function $f : \mathbb{R} \to \mathbb{R}$ with period 2π, that is, $f(x + 2\pi) = f(x)$, $x \in \mathbb{R}$. If f satisfies some regularity conditions (see, e.g., p. 480 in Kreyszig) we have

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx\right) \tag{1}$$

where (see, e.g., p. 476 in Kreyszig)

$$\begin{aligned} a_0 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx, \\ a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\, dx, \quad n \in \mathbb{N}, \\ b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\, dx, \quad n \in \mathbb{N}. \end{aligned}$$

Repetition: Fourier series

Consider a periodic function $f : \mathbb{R} \to \mathbb{R}$ with period $2L$, that is, $f(x + 2L) = f(x)$, $x \in \mathbb{R}$. If f satisfies some regularity conditions (see, e.g., p. 480 in Kreyszig) we have

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi}{L} x + b_n \sin \frac{n\pi}{L} x \right) \tag{2}$$

where (see, e.g., p. 484 in Kreyszig)

$$a_0 = \frac{1}{2L} \int_{-L}^{L} f(x)\, dx,$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi}{L} x\, dx, \quad n \in \mathbb{N},$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi}{L} x\, dx, \quad n \in \mathbb{N}.$$

Motivation of introducing the Fourier integral

So far, we have dealt with Fourier-series as a way to represent/approximate piece-wise continuous functions with well-defined one-sided derivatives on a finite interval (or if they are periodic) by a sum of trigonometric functions. As a next step, we will have a look on functions that are defined on the entire x-axis (and not necessarily periodic) which will lead to the concept of the Fourier transform which itself, as we shall see, is very similar (in its algebraic form) to the Laplace transform we have already discussed in detail.

Example

We start with an example of a function with period $2L$ and analyze its Fourier series as L approaches infinity. Therefore, let $f_L : \mathbb{R} \to \mathbb{R}$ be given by

$$f_L(x) = \begin{cases} 0 & \text{if } -L < x < -1 \text{ or } 1 < x < L \\ 1 & \text{if } -1 < x < 1 \end{cases}$$

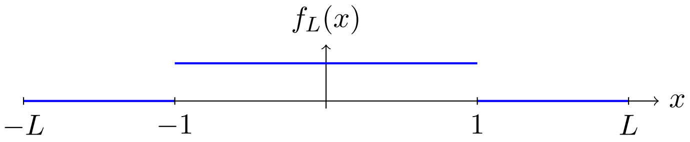
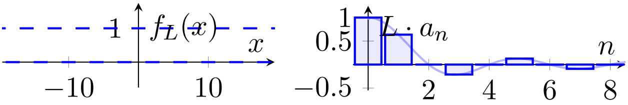


Figure: Function $f_L(x)$

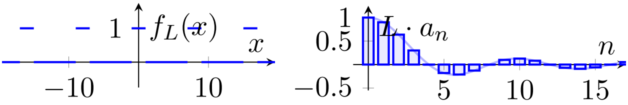
Example

We can now see what happens when the period $2L$ is increased. In the figure on the next slide, the functions $f_L(x)$ and their Fourier series coefficients a_n are depicted for $L \in \{2, 4, 8\}$. We see: the function $\sin(w)/w$ is the interesting mathematical structure behind the discrete numerical values of the Fourier coefficients. In the next subsection, this will be handled more formally and generally leading to the Fourier integral (as a step towards the Fourier transform later on).

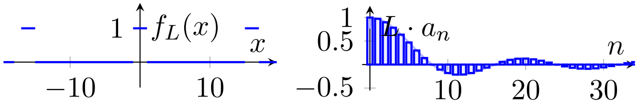
$L = 2$



$L = 4$



$L = 8$



$L \to \infty$

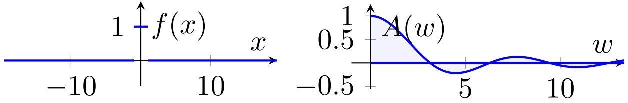


Figure: Amplitude spectra of the function f_L

From Fourier Series to Fourier Integral

For the more general case, we do now consider an (almost) arbitrary function $f_L(x)$ of period $2L$ that can be represented by a Fourier series

$$f_L(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi}{L} x + b_n \sin \frac{n\pi}{L} x\right).$$

To be precise, we will assume the following conditions on f_L:

From Fourier Series to Fourier Integral

1. For any finite value of L, the function f_L is piece-wise continuous and the one-sided derivatives exist in the closed interval $[-L, L]$.
2. The limit function $f(x) := \lim_{L\to\infty} f_L(x)$ is absolutely integrable. That means, the limits

$$\lim_{a\to-\infty} \int_a^0 |f(x)|\, dx\,, \quad \lim_{b\to\infty} \int_0^b |f(x)|\, dx$$

exist (and are finite).
3. The limit function $f(x)$ is also piece-wise continuous and its one-sided derivatives at any point exist.

Example

Coming back to the introducing example, we are looking for a Fourier integral representation of

$$f(x) = \begin{cases} 1 & \text{if } |x| < 1 \\ 0 & \text{else.} \end{cases}$$

We obtain

$$f(x) = \frac{2}{\pi} \cdot \int_0^\infty \frac{\sin(w) \cdot \cos(wx)}{w}\, dw.$$

Example (cont.)

If we now consider the point $x = 1$, the original function f is discontinuous and the value of the Fourier integral attains the average of left and right-side limit of f, that is $f(1) = \frac{1}{2} \cdot (0 + 1) = \frac{1}{2}$. The theorem therefore provides the following identities

$$\int_0^\infty \frac{\sin(w)\cos(wx)}{w}\,dw = \begin{cases} \pi/2 & \text{if } 0 \leq x < 1 \\ \pi/4 & \text{for } x = 1 \\ 0 & \text{for } x > 1. \end{cases}$$

Example (cont.)

In the special case $x = 0$, we have $\cos(wx) = 1$ and so

$$\frac{\pi}{2} = \lim_{u\to\infty} \underbrace{\int_0^u \frac{\sin(w)}{w}\, dw}_{=:\mathrm{Si}(u)} .$$

The involved integral is the limit case of the so-called **sine-integral** $\mathrm{Si}(u)$ and a prominent example of a function that is only given by an integral that cannot be explicitly evaluated. The sine-integral was studied by Joseph Liouville in the 1800's and comes up in connection with the Wilbraham-Gibbs-constant, the Euler-Mascheroni constant and the geometric form of a clothoid (Euler-spiral).

Example (cont.)

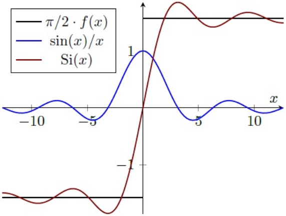


Figure: The sine integral, its integrand and the limit case $\pi/2 \cdot f(x)$

Example (cont.)

The Fourier integral representation of a function is an improper integral. So, as truncating the Fourier series yields approximations to a function, we can also work with proper integrals instead to approximate functions. In the above example, the sine integral can be used to approximate the originally given function $f(x)$ above.

Example (cont.)

In the figure below the approximations to $f(x)$ by means of

$$f(x) \approx \frac{2}{\pi} \cdot \int_0^u \frac{\cos(wx)\sin(w)}{w}\, dw =: \tilde{f}(x;u)$$

are shown for $u \in \{8, 16, 32\}$. We see that close to the points of discontinuity the approximations oscillate heavily (Gibbs phenomenon).

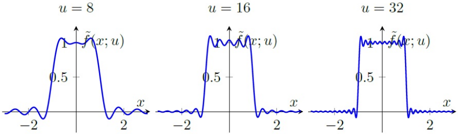


Figure: Illustration of the Gibbs phenomenon

Example (cont.) - Gibbs phenomenon

To explain this, we use the trigonometric identity

$$\sin(a) \cdot \cos(b) = \frac{1}{2} \cdot (\sin(a+b) + \sin(a-b))$$

to express $\tilde{f}(x;u)$ as

$$\tilde{f}(x;u) = \frac{1}{\pi} \cdot \int_0^u \frac{\sin(w + wx)}{w}\, dw + \frac{1}{\pi} \cdot \int_0^u \frac{\sin(w - wx)}{w}\, dw\,.$$

Example (cont.) - Gibbs phenomenon

Setting $t := w + wx$ (so $\frac{dt}{dw} = 1 + x = \frac{t}{w}$) and $\tilde{t} := w - wx$ (so $\frac{d\tilde{t}}{dw} = 1 - x = \frac{\tilde{t}}{w}$) and using that the sine function is odd, we get

$$\begin{aligned}\tilde{f}(x;u) &= \frac{1}{\pi} \cdot \int_0^{(x+1)u} \frac{\sin t}{t}\, dt - \frac{1}{\pi} \cdot \int_0^{(x-1)u} \frac{\sin(\tilde{t})}{\tilde{t}}\, d\tilde{t} \\ &= \frac{1}{\pi} \cdot (\mathrm{Si}((x+1)u) - \mathrm{Si}((x-1)u)) \ .\end{aligned}$$

So, the larger u is chosen the better the approximation gets but the maximum errors of that approximation remain the same and just get pushed towards the points of discontinuity. (Corresponding to a scaling of the u-axis in $\mathrm{Si}(u)$)