NTNU | Kunnskap for en bedre verden

TMA4130 MATEMATIKK 4N

Lecture 20: Fourier Analysis: Fourier Series, Fourier Integral

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Important

If you develop respiratory symptoms, please stay home!

Plan for today

1. We complete the topic on approximation by trigonometric polynomials from Lecture 19
2. Fourier Integral

Example (Sawtooth Wave Function)

The error for N is:

N	E^*
1	8.104480505840698
2	4.962887852250915
3	3.566624450655453 4
4	2.781226287258008
5	2.278571462683629 5
6	1.929505612284771 1
7	1.673049069134577 8
8	1.476699528285209 3
9	1.321559150330159 6
10	1.195895444186575 7
20	0.612872236171043
30	0.411975256435482 6
100	0.125037481966103 31
1000	0.012560089523418 583

Repetition: Fourier series

Consider a periodic function $f : \mathbb{R} \to \mathbb{R}$ with period 2π, that is, $f(x + 2\pi) = f(x)$, $x \in \mathbb{R}$. If f satisfies some regularity conditions (see, e.g., p. 480 in Kreyszig) we have

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx\right) \tag{1}$$

where (see, e.g., p. 476 in Kreyszig)

$$\begin{aligned} a_0 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx, \\ a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\, dx, \quad n \in \mathbb{N}, \\ b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\, dx, \quad n \in \mathbb{N}. \end{aligned}$$

Repetition: Fourier series

Consider a periodic function $f : \mathbb{R} \to \mathbb{R}$ with period $2L$, that is, $f(x + 2L) = f(x)$, $x \in \mathbb{R}$. If f satisfies some regularity conditions (see, e.g., p. 480 in Kreyszig) we have

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi}{L} x + b_n \sin \frac{n\pi}{L} x \right) \tag{2}$$

where (see, e.g., p. 484 in Kreyszig)

$$a_0 = \frac{1}{2L} \int_{-L}^{L} f(x)\,dx,$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi}{L} x\,dx, \quad n \in \mathbb{N},$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi}{L} x\,dx, \quad n \in \mathbb{N}.$$

Motivation of introducing the Fourier integral

So far, we have dealt with Fourier-series as a way to represent/approximate piece-wise continuous functions with well-defined one-sided derivatives on a finite interval (or if they are periodic) by a sum of trigonometric functions. As a next step, we will have a look on functions that are defined on the entire x-axis (and not necessarily periodic) which will lead to the concept of the Fourier transform which itself, as we shall see, is very similar (in its algebraic form) to the Laplace transform we have already discussed in detail.

Example

We start with an example of a function with period $2L$ and analyze its Fourier series as L approaches infinity. Therefore, let $f_L : \mathbb{R} \to \mathbb{R}$ be given by

$$f_L(x) = \begin{cases} 0 & \text{if } -L < x < -1 \text{ or } 1 < x < L \\ 1 & \text{if } -1 < x < 1 \end{cases}$$

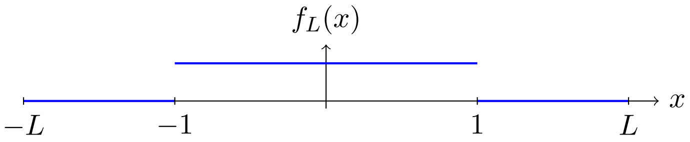
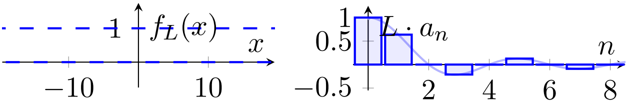


Figure: Function $f_L(x)$

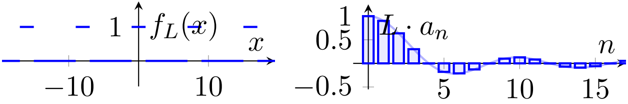
Example

We can now see what happens when the period $2L$ is increased. In the figure on the next slide, the functions $f_L(x)$ and their Fourier series coefficients a_n are depicted for $L \in \{2, 4, 8\}$. We see: the function $\sin(w)/w$ is the interesting mathematical structure behind the discrete numerical values of the Fourier coefficients. In the next subsection, this will be handled more formally and generally leading to the Fourier integral (as a step towards the Fourier transform later on).

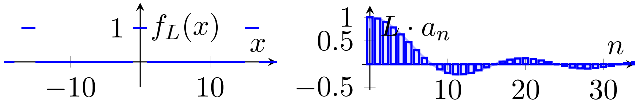
$L = 2$



$L = 4$



$L = 8$



$L \to \infty$

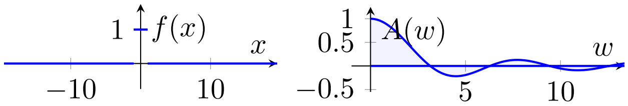


Figure: Amplitude spectra of the function f_L

References

The material of this lecture was based on Chapter 11.4 and 11.7 of the book

Advanced Mathematical Engineering by Erwin Kreyszig (John Wiley & Sons, 10th edition, 2011).

Moreover, we recommend the lecture notes by Morten Nome (in Norwegian), who taught the 2019 edition of this course. You can Morten's lecture notes on Fourier series here:

https://www.math.ntnu.no/emner/TMA4125/2019v/notater/02-fourierrekker.pdf.

Next Lecture

- We continue with the Fourier integral
- Fourier transform