Lecture 19

- If $f(x)$ is an odd function, the Fourier series of $f(x)$ reduces to a Fourier sine series, that is,

$$f(x) = \sum_{n=1}^{\infty} b_n \cdot \sin \frac{n\pi}{L} x \qquad (f \text{ odd})$$

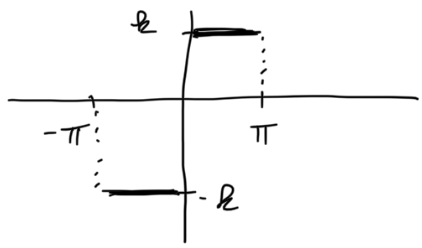
with coefficients

$$b_n = \frac{2}{L} \int_0^L \underbrace{\underbrace{f(x)}_{\text{odd}} \cdot \underbrace{\sin \frac{n\pi}{L} x}_{\text{odd}}}_{\text{even}} \, dx$$

Note: $a_0 = \frac{1}{2L} \int_{-L}^{L} f(x)\, dx = 0$ as f is odd,

$$a_n = \frac{1}{L} \int_{-L}^{L} \underbrace{\underbrace{f(x)}_{\text{odd}} \cdot \underbrace{\cos \frac{n\pi x}{L}}_{\text{even}}}_{\text{odd}} \, dx = 0.$$

Example: In the "basic example" in Lecture 17, the rectangular wave function is odd:



$\Rightarrow$ Its Fourier series is a Fourier sine series.

Theorem (Sum and Scalar Multiple)

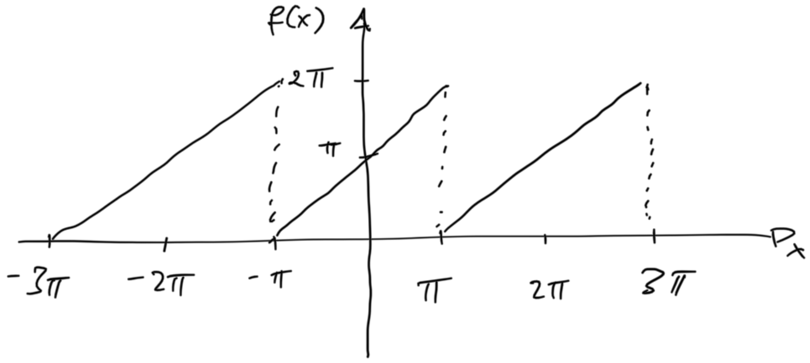
Theorem (sum and scalar multiple)

The Fourier coefficients of a sum $f_1 + f_2$ are the sums of the corresponding Fourier coefficients of f_1 and f_2.

The Fourier coefficients of $c \cdot f$ are c times the corresponding Fourier coefficients of f.

Example: Let $f(x) = x + \pi$ if $-\pi < x < \pi$, and $f(x+2\pi) = f(x)$.

$(p = 2L = 2\pi \Rightarrow L = \pi)$.



We have $f = f_1 + f_2$ with $f_1(x) = x$, $f_2(x) = \pi$.

- $f_1(x) = x$ is odd. Therefore, its Fourier series is a Fourier sine series with coefficients:

$$b_n = \frac{2}{\pi} \int_0^L f_1(x) \cdot \sin \frac{n\pi x}{L} \, dx$$

$$= \frac{2}{\pi} \int_0^\pi x \sin nx \, dx$$

integration by parts

$$= \frac{2}{\pi} \left[-\frac{x}{n} \cos nx \right]_0^\pi - \left(-\frac{2}{\pi} \int_0^\pi \frac{1}{n} \cos nx \, dx \right)$$

$$= \frac{2}{\cancel{\pi}} \left(-\frac{\cancel{\pi}}{n} \cos n\pi + 0 \right) + \frac{2}{\pi n} \cdot \left[\frac{1}{n} \cdot \sin nx \right]$$

$$\underbrace{\qquad\qquad}_{=0}$$

$$= -\frac{2}{n}\cos n\pi$$

$$= \pm\frac{2}{n} \left(\begin{array}{ll} +\frac{2}{n} & \text{for odd } n \\ -\frac{2}{n} & \text{for even } n \end{array}\right)$$

$$\Rightarrow b_1 = 2, \quad b_2 = -\frac{2}{2} = -1, \quad b_3 = \frac{2}{3}, \quad b_4 = -\frac{2}{4} = -$$

- $f_2 = \pi$ is constant. Hence, its Fourier series ... is also the constant $a_0 = \pi$.

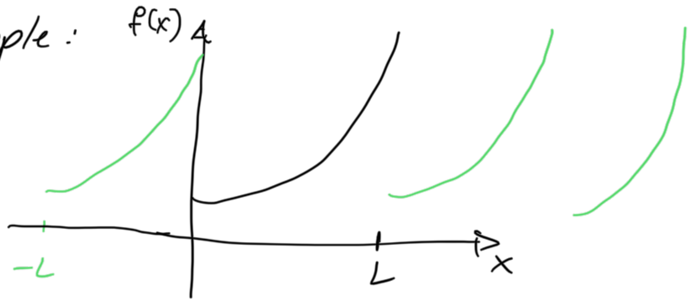
$\Rightarrow$ The Fourier series of $f(x) = x + \pi$ is

$$f(x) = \pi + 2\left(\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \frac{1}{4}\sin 4x + \dots\right).$$

Half-Range Expansions

Goal: Represent a function by a Fourier series.

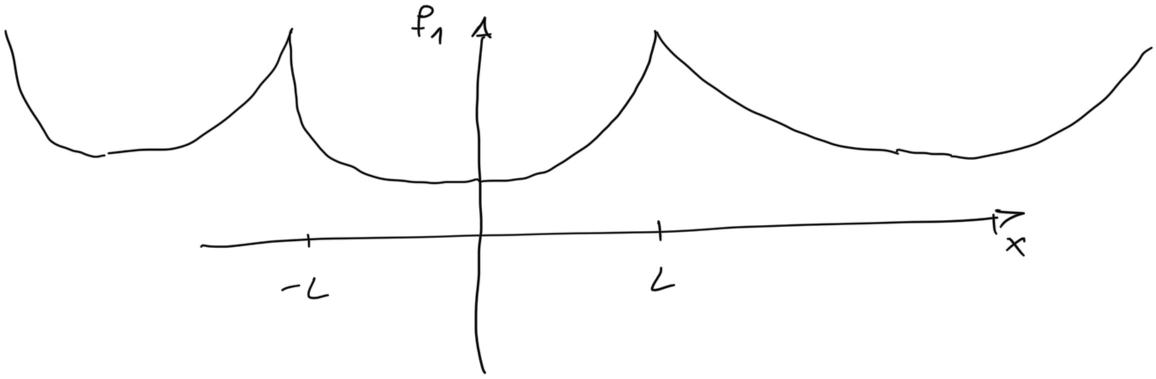
For example:



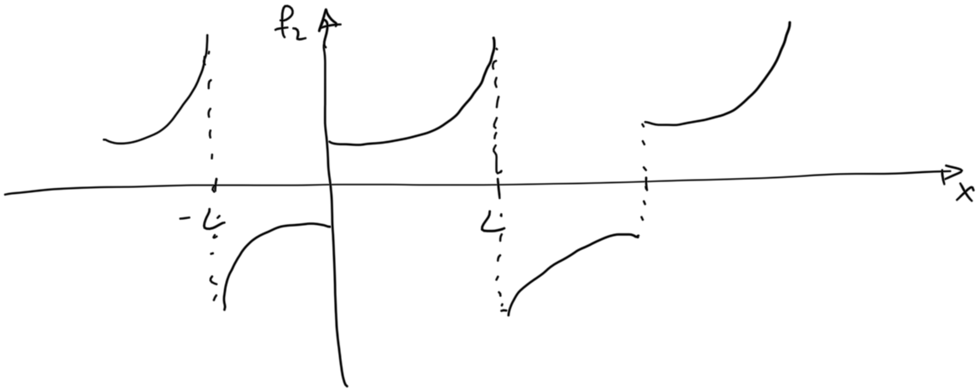
But: This extension contains both sine and cosine terms

Better: Use only sine or cosine terms.

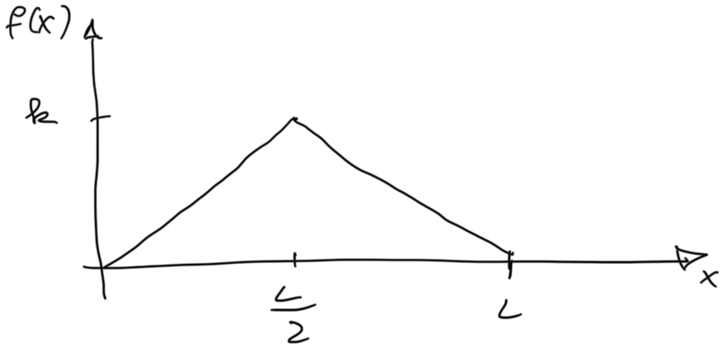
We consider the <u>even periodic extension</u> f_1 of f:



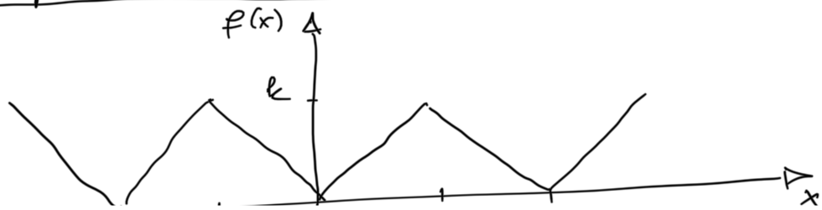
or: Consider the <u>odd periodic extension</u> f_2 of f

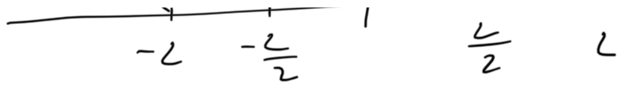


<u>Example</u>: let $f(x) = \begin{cases} \frac{2k\,x}{L} & \text{if } 0 < x < \frac{L}{2} \\ \frac{2k}{L}(L-x) & \text{if } \frac{L}{2} < x < L \end{cases}$



- <u>Even periodic extension</u>:





$$a_0 = \frac{1}{L}\int_0^L f(x)\,dx$$

$$= \frac{1}{L}\left(\int_0^{L/2} \frac{2kx}{L}\,dx + \int_{L/2}^{L} \frac{2k}{L}(L-x)\,dx \right)$$

$$= \frac{1\cdot 2k}{L^2}\left(\int_0^{L/2} x\,dx + \int_{L/2}^{L} (L-x)\,dx \right)$$

$$= \frac{2k}{L^2}\left(\left[\frac{1}{2}x^2\right]_0^{L/2} + \left[Lx - \frac{1}{2}x^2\right]_{L/2}^{L} \right)$$

$$= \frac{2k}{L^2}\left(\underline{\frac{1}{2}\cdot\frac{L^2}{4}} - 0 + \cancel{L^2} - \cancel{\frac{1}{2}L^2} - \cancel{\frac{L^2}{2}} + \underline{\frac{1}{8}L^2} \right)$$

$$= \frac{2k}{\cancel{L^2}}\cdot\frac{1}{4}\cancel{L^2} = \underline{\underline{\frac{1}{2}k}}$$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx$$

$$= \frac{2}{L}\cdot\frac{2k}{L}\left(\int_0^{L/2} x\cos\frac{n\pi x}{L}\,dx + \int_{L/2}^{L} (L-x)\cos\frac{n\pi x}{L}\,dx \right)$$

$$= \frac{4k}{L^2}\left(\quad - \quad '' \quad - \quad \right)$$

· First integral:

$$\underline{\int_0^{L/2} x\cos\frac{n\pi x}{L}\,dx} \overset{\substack{\text{Integration}\\ \text{by parts}}}{=} \left[\frac{L}{n\pi}x\sin\frac{n\pi x}{L}\right]_0^{\frac{L}{2}} - \int^{\frac{L}{2}} \frac{L}{n\pi}\sin\frac{n\pi x}{L}$$

$$= \frac{L}{n\pi}\cdot\frac{L}{2}\cdot\sin\frac{n\pi \cancel{L}}{\cancel{L}\cdot 2} - 0 + \left[\frac{L^2}{n^2\pi^2}\cos\frac{n\pi x}{L}\right]_0^{L/2}$$

$$= \frac{L^2}{2n\pi}\cdot\sin\frac{n\pi}{2} + \frac{L^2}{n^2\pi^2}\cos\frac{n\pi L}{2L} - \frac{L^2}{n^2\pi^2}\cdot 1$$

$$= \frac{L^2}{2n\pi}\sin\frac{n\pi}{2} + \frac{L^2}{n^2\pi^2}\left(\cos\frac{n\pi}{2} - 1\right)$$

· Second Integral:

$$\int_{L/2}^{L}(L-x)\cos\frac{n\pi x}{L}\,dx = L\int_{L/2}^{L}\cos\frac{n\pi x}{L}\,dx - \int_{L/2}^{L}x\cos\frac{n\pi x}{L}\,dx$$

$$= L\cdot\frac{L}{n\pi}\left[\sin\frac{n\pi x}{L}\right]_{L/2}^{L} - \left[\frac{L}{n\pi}x\sin\frac{n\pi x}{L}\right]_{L/2}^{L} + \int_{L/2}^{L}\frac{L}{n\pi}\sin\frac{n\pi x}{L}$$

$$= \frac{L^2}{n\pi}\Big(\underbrace{\sin(n\pi)}_{=0} - \sin\frac{n\pi}{2}\Big) - \frac{L}{n\pi}L\underbrace{\sin n\pi}_{=0} + \frac{L^2}{2n\pi}\sin\frac{n\pi}{2}$$

$$- \left[\frac{L}{n\pi}\frac{L}{n\pi}\cos\frac{n\pi x}{2}\right]_{L/2}^{L}$$

$$= -\frac{L^2}{n\pi}\sin\frac{n\pi}{2} + \frac{L^2}{2n\pi}\sin\frac{n\pi}{2} - \frac{L^2}{n^2\pi^2}\cos n\pi + \frac{L^2}{n^2\pi^2}\cos\frac{n\pi}{2}$$

Inserting both integrals in an yields:

$$a_n = \frac{4\cancel{k}}{\cancel{L^2}}\left(\frac{\cancel{L^2}}{2n\pi}\sin\frac{\pi n}{2} + \frac{\cancel{L^2}}{n^2\pi^2}\left(\cos\frac{n\pi}{2} - 1\right)\right)$$

$$-\frac{\cancel{k}^{\,2}}{n\pi}\sin\frac{n\pi}{2} + \frac{\cancel{k}^{\,2}}{2n\pi}\sin\frac{n\pi}{2} - \frac{\cancel{k}^{\,2}}{n^2\pi^2}\cos n\pi + \frac{\cancel{k}^{\,2}}{n^2\pi^2}\cos$$

$$= 4k\left(\frac{1}{n^2\pi^2}\left(\cos\frac{n\pi}{2} - 1\right) - \frac{1}{n^2\pi^2}\cos n\pi + \frac{1}{n^2\pi^2}\cos\frac{n\pi}{2}\right.$$

$$= \frac{4k}{n^2\pi^2}\left(2\cos\frac{n\pi}{2} - 1 - \cos n\pi\right)$$

Thus,

$$a_0 = \frac{k}{2}$$

$$a_1 = \frac{4k}{\pi^2}\left(2\cdot 0 - 1 - (-1)\right) = 0$$

$$a_2 = \frac{\cancel{4}k}{\cancel{4}\pi^2}\left(2\cdot\underbrace{\cos\frac{\cancel{2}\pi}{\cancel{2}}}_{=-1} - 1 - \underbrace{\cos 2\pi}_{=1}\right)$$

$$= \frac{k}{\pi^2}\left(-2 - 1 - 1\right) = -\frac{4k}{\pi^2} = -\frac{16k}{2^2\pi^2}$$

$$a_3 = a_4 = a_5 = 0$$

$$a_6 = \frac{4k}{36\pi^2}\left(2\underbrace{\cos\frac{6\pi}{2}}_{=-1} - 1 - \underbrace{\cos 6\pi}_{=1}\right)$$

$$= -\frac{4k\cdot 4}{36\pi^2} = -\frac{16k}{6^2\pi^2}$$

$$a_7 = a_8 = a_9 = 0$$

$$a_{10} = -\frac{16k}{10^2\pi^2}.$$

Hence, the even periodic extension of $f(x)$ is

$$\frac{k}{2} - \frac{16k}{\pi^2}\left(\frac{1}{2^2}\cos\frac{2\pi}{L}x + \frac{1}{6^2}\cos\frac{6\pi x}{L} + \dots\right)$$

Approximation by Trigonometric Polynomials

Approximation theory: Goal: approximate functions by other, simple functions.

Let $f(x)$ be a function on the interval $-\pi \le x \le \pi$ that can be represented by a Fourier series.

Then the Nth partial sum

$$S_N(x) := a_0 + \sum_{n=1}^{N}(a_n\cos nx + b_n\sin nx)$$

is an "approximation" of $f(x)$, and we write

$$S_N(x) \approx f(x).$$

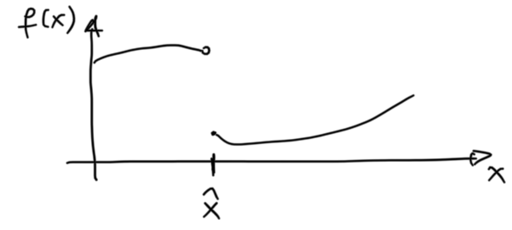
Let N be fixed. Question: Is S_N the "best" approximation of by a trigonometric polynomial of degree N?

Let such a trigonometric polynomial be denoted by:

$$F(x) = A_0 + \sum^{N}(A_n\cos nx + B_n\sin nx).$$

$n=1$

- "best" approximation means "small error".
- Possible definitions of the error:
  - $E = \sup_x | f(x) - F(x) |$



- $$E = \int_{-\pi}^{\pi} | f(x) - F(x) | \, dx$$

Better: • $E = \int_{-\pi}^{\pi} \left(f(x) - F(x) \right)^2 dx$ is called "square error" of F relative to the function f on the interval $[-\pi, \pi]$

- Clearly, $E \geq 0$.

We show now that minimizing the error leads to a function F which equals S_N.

We get:

$$E = \int_{-\pi}^{\pi} \left(f(x) - F(x) \right)^2 dx = \int_{-\pi}^{\pi} \left(f^2(x) - 2 f(x) F(x) + F^2(x) \right) dx$$

$$= \int_{-\pi}^{\pi} f^2 \, dx - 2 \underbrace{\int_{-\pi}^{\pi} f F \, dx} + \underline{\int_{-\pi}^{\pi} F^2 \, dx}.$$

Let us insert $\underline{F(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos nx + B_n \sin nx}$

into E :

- Second integral :

$$\int_{-\pi}^{\pi} f(x) \cdot \left(A_0 + \sum_{n=1}^{\infty} A_n \cos nx + B_n \sin nx\right) dx$$

$$\approx \int_{-\pi}^{\pi} \left(a_0 + a_1 \cos x + b_1 \sin x + \ldots + a_N \cos Nx + b_N \sin Nx\right) \cdot \left(A_0 + A_1 \cos x + B_1 \sin x + \ldots + A_N \cos Nx + B_N \sin Nx\right) dx$$

$$= \pi \left(2 A_0 a_0 + A_1 a_1 + \ldots + A_N a_N + B_1 b_1 + \ldots + B_N b_N\right)$$

because :

- $\int_{-\pi}^{\pi} a_0 A_0 \, dx = a_0 A_0 \cdot 2\pi$
- $\int_{-\pi}^{\pi} (\cos nx)^2 \, dx = \int_{-\pi}^{\pi} (\sin nx)^2 \, dx = \pi$
- $\int_{-\pi}^{\pi} (\cos nx)(\sin mx) = 0$ if $n \neq m$,

  and $\int_{-\pi}^{\pi} \cos nx \, dx = \int_{-\pi}^{\pi} \sin nx = 0$.