NTNU | Kunnskap for en bedre verden

TMA4130 MATEMATIKK 4N

Lecture 12: Convolution. Integral Equations. Differentiation and Integration of Transforms. Systems of ODEs

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Repetition: Laplace Transform

Definition: Laplace transform

Given a function $f(t)$ ($t =$ time, $f : \mathbb{R}_+ \to \mathbb{R}$), its **Laplace transform** is defined as

$$F(s) := \mathcal{L}(f) := \int_0^\infty e^{-st} f(t)dt.$$

Theorem: Linearity of the Laplace Transform

The Laplace transform is a linear operation; that is, for any functions f and g whose transforms exist and any constants a and b the transform of $af(t) + bg(t)$ exists, and

$$\mathcal{L}\{af(t) + bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}.$$

$$\mathcal{L}\{f(t) + g(t)\} = \mathcal{L}\{f(t)\} + \mathcal{L}\{g(t)\}$$

$$\mathcal{L}\{f(t) \cdot g(t)\} \neq \mathcal{L}\{f(t)\} \cdot \mathcal{L}\{g(t)\}$$

is the Laplace transform of the convolution of f and g

Convolution

Definition: Convolution of two functions f and g

The convolution of two functions f and g is defined by

$$(f * g)(t) := \int_0^t f(\tau)g(t - \tau)d\tau.$$

Convolution

Convolution Theorem

Let $f(t)$ and $g(t)$ be defined and piecewise continuous on every finite interval on the semi-axis $t \geqq 0$ and suppose that[1] holds for all $t \geqq 0$ and some constants M, k, $\overline{M}$ and $\overline{k}$, such that the Laplace transforms $\mathcal{L}(f)$ and $\mathcal{L}(g)$ exist. Then the product $H := FG$ is the transform of the function

$\mathcal{L}\{h\} = \mathcal{L}\{f\} \cdot \mathcal{L}\{g\}$

$$h(t) = (f * g)(t) := \int_0^t f(\tau)g(t-\tau)d\tau.$$

[1] $\exists\ M, k\ s.t.\ \forall t \geqq 0 :\ |f(t)| \leqq Me^{kt}$ and $\exists\ \overline{M}, \overline{k}\ s.t.\ \forall t \geqq 0 :\ |g(t)| \leqq \overline{M}e^{\overline{k}t}$.

Proof: $F(s) = \int_0^\infty e^{-s\tau} f(\tau)\, d\tau$, $G(s) = \int_0^\infty e^{-sp} g(p)\, dp$

Now set $t = p + \tau \Rightarrow p = t - \tau \Rightarrow dp = dt$

$p = 0 \Rightarrow t = \tau$

$$\Rightarrow G(s) = \int_\tau^\infty e^{-s(t-\tau)} g(t-\tau)\, dt$$

$$= e^{s\tau} \int_\tau^\infty e^{-st} g(t-\tau)\, dt$$

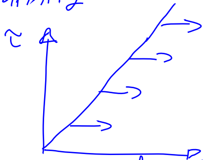
$$\Rightarrow F(s) \cdot G(s) = \int_0^\infty e^{-s\tau} f(\tau)\, e^{s\tau} \int_\tau^\infty e^{-st} g(t-\tau)\, dt\, d\tau$$

$e^{s\tau} \int_\tau^\infty e^{-st} g(t-\tau)\, dt = G(s)$

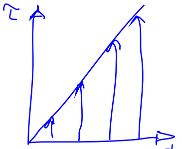
$$= \int_0^\infty f(\tau) \int_\tau^\infty e^{-st} g(t-\tau)\, dt\, d\tau$$

we integrate over τ from 0 to ∞

first, we integrate for fixed τ over t from τ to infinity



t goes from τ to ∞



τ goes from zero to t

$$F(s)\cdot G(s) = \int_0^\infty e^{-st} \int_0^t f(\tau)\, g(t-\tau)\, d\tau\, dt$$

$$= \int_0^\infty e^{-st} \cdot h(t)\, dt$$

$$= \mathcal{L}\{h(t)\} = H(s)$$

Example

Let $H(s) = \frac{1}{(s-a)s}$. Find $h(t)$.

Example

Let $H(s) = \frac{1}{(s^2+\omega^2)^2}$. Find $h(t)$.

Example: Unusual Properties of Convolution

- $f * 1 \neq f$ (in general):

- $(f * f)(t) \geq 0$ may not hold:

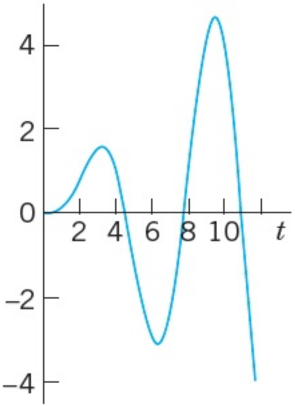


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Figure: $(f * f)(t) \ngeq 0.$

Example: Repeated Complex Factors. Resonance

Application to Nonhomogeneous Linear ODEs

Integral Equations

Example: A Volterra Integral Equation of the Second Kind

Differentiation and Integration of Transforms

Systems of ODEs

References

The material of this lecture was based on Chapter 6.5–6.7 of the book

Advanced Mathematical Engineering by Erwin Kreyszig (John Wiley & Sons, 10th edition, 2011)

and Chapter 6 in

Differential Equations Demystified by Steven G. Krantz (McGraw-Hill, 2005).

Moreover, we recommend the lecture notes by Morten Nome (in Norwegian), who taught the 2019 edition of this course. You can download Lecture 1 of Morten's lecture notes collection here:

https://www.math.ntnu.no/emner/TMA4125/2019v/notater/01-laplacetransform.pdf

Next lectures

- Numerical methods for ordinary differential equations