NTNU | Kunnskap for en bedre verden

TMA4130 MATEMATIKK 4N

Lecture 11: Unit Step Function (Heaviside Function). Second Shifting Theorem (t-Shifting). Dirac's Delta Function.

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Unit Step Function (Heaviside Function) $u(t-a)$

Definition: Unit Step Function (Heaviside Function) $u(t-a)$

Let $a \geqq 0$. The **unit step function** (or **Heaviside function**) $u(t-a)$ is defined as follows:

$$u(t-a) := \begin{cases} 0 & (t < a) \\ 1 & (t > a) \end{cases}$$

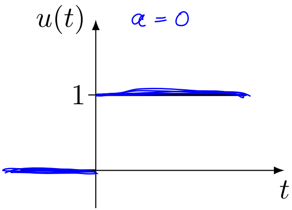

Figure: The unit step function $u(t)$ for $a = 0$ (with jump at 0).

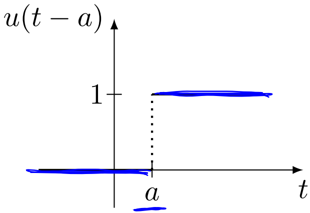

Figure: The unit step function $u(t-a)$ (with jump at a).

Shifted Functions

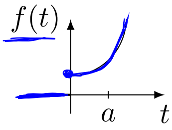

Figure: Function $f(t)$ with $f(t) = 0$ for $t < 0$.

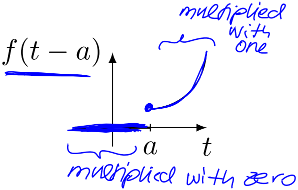

Figure: Function $f(t-a)$.

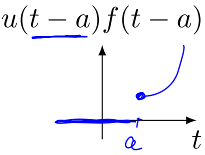

Figure: Function $u(t-a)f(t-a)$.

$f(t-a) = f(a-a) = f(0)$ for $t = a$

Shifted and Switched On/Off-Functions

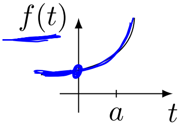

Figure: Function $f(t)$.

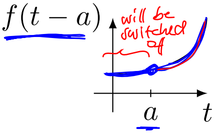

Figure: Function $f(t-a)$.

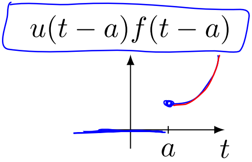

Figure: Function $u(t-a)f(t-a)$.

Laplace Transform of the unit step function

$$\mathcal{L}\{u(t-a)\} = \int_0^\infty e^{-st} u(t-a)\, dt$$

$$= \int_a^\infty e^{-st} \cdot 1\, dt$$

$$= \left[-\frac{1}{s} \cdot e^{-st} \right]_a^\infty$$

$$= 0 - \left(-\frac{1}{s} \cdot e^{-s \cdot a} \right)$$

$$= \frac{1}{s} \cdot e^{-sa}$$

Theorem: Second Shifting Theorem; Time Shifting

If $f(t)$ has the transform $F(s)$, then the function

$$\tilde{f}(t) = f(t-a)u(t-a) := \begin{cases} 0 & (t < a) \\ f(t-a) & (t > a) \end{cases}$$

has the transform $e^{-as}F(s)$. That is, if $\mathcal{L}(f) = F(s)$, then

$$\mathcal{L}\{f(t-a)u(t-a)\} = e^{-as}F(s).$$

Or, if we take the inverse on both sides, we can write

$$f(t-a)u(t-a) = \mathcal{L}^{-1}\left\{e^{-as}F(s)\right\}.$$

Proof: $e^{-as} \cdot F(s) = e^{-as} \cdot \int_0^\infty e^{-s\tau} \cdot f(\tau)\, d\tau$

$$= \int_0^\infty e^{-s(\tau + a)} f(\tau)\, d\tau$$

Substitution: $t := \tau + a \implies \tau = t - a \;,\; dt = d\tau$

$\tau = 0 \implies t = a$

$$\implies e^{-as} F(s) = \int_a^\infty e^{-st} \cdot f(t-a)\, dt$$

$$= \int_0^\infty e^{-st} \cdot \underbrace{u(t-a) \cdot f(t-a)}_{=: \tilde{f}(t)}\, dt$$

$$= \int_0^\infty e^{-st} \tilde{f}(t)\, dt .$$

■

Example

Write the following function using unit step functions and find its transform.

$$f(t) = \begin{cases} 2 & (0 < t < 1) \\ \frac{1}{2}t^2 & (1 < t < \frac{1}{2}\pi) \\ \cos t & (t > \frac{1}{2}\pi). \end{cases}$$

Example

Find the inverse transform $f(t)$ of

$$F(s) = \frac{e^{-s}}{s^2+\pi^2} + \frac{e^{-2s}}{s^2+\pi^2} + \frac{e^{-3s}}{(s+2)^2}.$$

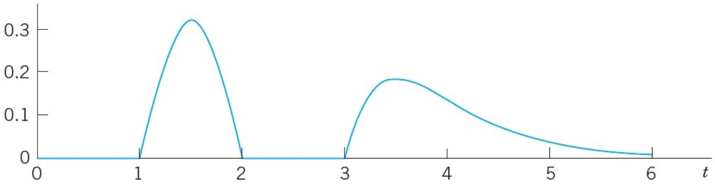

Figure: $f(t) = \frac{1}{\pi}\sin(t(t-1))u(t-1) + \frac{1}{\pi}\sin(t(t-2))u(t-2).$

Dirac's Delta Function

Consider the function

$$f_k(t-a) = \begin{cases} \frac{1}{k} & (a \leqq t \leqq a+k) \\ 0 & (\text{else}). \end{cases}$$

This function represents, for instance, a force of magnitude $\frac{1}{k}$ acting from $t = a$ to $t = a + k$, where k is positive and small. In mechanics, the integral of a force acting over a time interval $a \leqq t \leqq a + k$ is called the **impulse** of the force.

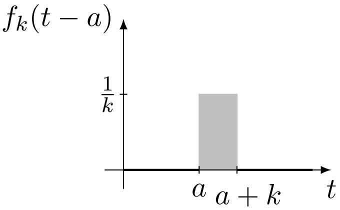

Figure: Function $f_k(t-a)$.

Dirac's Delta Function

The impulse of f_k is

$$I_k := \int_o^{\infty} f_k(t-a)dt = \int_a^{a+k} \frac{1}{k}dt =$$

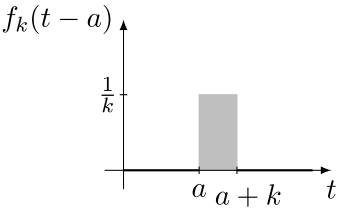

Figure: Function $f_k(t-a)$.

Dirac's Delta Function

Consider $k \to 0$:

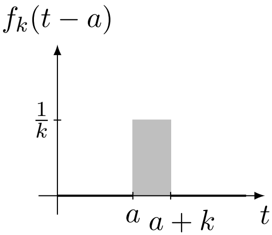

Figure: Function $f_k(t-a)$.

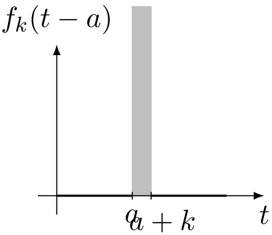

Figure: Function $f_k(t-a)$.

Dirac's Delta Function

Definition: Dirac's Delta Function (Unit Impulse Function) $\delta(t-a)$

Dirac's delta function is defined as

$$\delta(t-a) := \lim_{k \to 0} f_k(t-a) = \begin{cases} \infty & (t=a) \\ 0 & (\text{else}). \end{cases}$$

Remark

Furthermore, we define $\int_0^\infty \delta(t-a)dt = 1$. However, from calculus we know that a function which is everywhere 0 except at a single point must have the integral equal to 0. Therefore, $\delta(t-a)$ is not a function in the ordinary sense as used in calculus, but a so-called generalized function.

Sifting Property of Dirac's Delta Function

Given a *continuous* function $g(t)$, it holds

$$\int_0^\infty g(t)\delta(t-a)dt = g(a).$$

Laplace Transform of Dirac's Delta Function

Example

Determine the response of the damped mass-spring system under a square wave, modeled by

$$y'' + 3y' + 2y = r(t) = u(t-1) - u(t-2), \quad y(0) = 0, \quad y'(0) = 0.$$

Example

Find the response of the system in the previous example with the square wave replaced by a unit impulse at time $t = 1$, that it, determine the response of the damped mass-spring system, modeled by

$$y'' + 3y' + 2y = r(t) = \delta(t-1), \quad y(0) = 0, \quad y'(0) = 0.$$

References

The material of this lecture was based on Chapter 6.3 and 6.4 of the book

Advanced Mathematical Engineering by Erwin Kreyszig (John Wiley & Sons, 10th edition, 2011).

Moreover, we recommend the lecture notes by Morten Nome (in Norwegian), who taught the 2019 edition of this course. You can download Lecture 1 of Morten's lecture notes collection here:

https://www.math.ntnu.no/emner/TMA4125/2019v/notater/01-laplacetransform.pdf

Next lecture

We continue with the Laplace transform by considering:

- Convolution
- Integral equations