NTNU | Kunnskap for en bedre verden

TMA4130 MATEMATIKK 4N

Lecture 10: Transforms of Derivatives and Integrals. ODEs

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Plan for the day

After today's lecture, you should be familiar with

- the Laplace transform of derivatives and integrals
- solving ODEs with the Laplace transform.

Solving linear ODEs and related initial value problems

The Laplace transform is a method of solving ODEs and initial value problems. The crucial idea is that operations of calculus on functions are replaced by operations of algebra on transforms. Roughly, differentiation of $f(t)$ will correspond to multiplication of $\mathcal{L}(f)$ by s and integration of $f(t)$ to division of $\mathcal{L}(f)$ by s. To solve ODEs, we must first consider the Laplace transform of derivatives.

Theorem: Laplace Transform of Derivatives

The transforms of the first and second derivatives of $f(t)$ satisfy

$$\mathcal{L}(f') = s\mathcal{L}(f) - f(0) \quad (1)$$

$$\mathcal{L}(f'') = s^2\mathcal{L}(f) - sf(0) - f'(0). \quad (2)$$

Formula (1) holds if $f(t)$ is continuous for all $t \geqq 0$ and satisfies the growth restriction[1] and $f'(t)$ is piecewise continuous on every finite interval on the semi-axis $t \geqq 0$. Similarly, (2) holds if f and f' are continuous for all $t \geqq 0$ and satisfy the growth restriction and f'' is piecewise continuous on every finite interval on the semi-axis $t \geqq 0$.

[1] $\exists\ M, k\ s.t.\ \forall t \geqq 0 : \ |f(t)| \leqq Me^{kt}$.

Proof: • Assume, in addition, that f' is continuous. Then, by integration by parts:

$$\mathcal{L}(f') = \int_0^\infty e^{-st} \cdot f'(t)\, dt$$

f cont.

$$= \left[e^{-st} f(t) \right]_0^\infty - \int_0^\infty \left(-s e^{-st}\right) \cdot f(t)\, dt$$

$$= \left[e^{-st} f(t) \right]_0^\infty + s \cdot \underbrace{\int_0^\infty e^{-st} f(t)\, dt}_{\mathcal{L}(f)}$$

$$= s \cdot \mathcal{L}(f) - f(0)$$

• $|e^{-st} f(t)| \leq e^{-st} \cdot |f(t)| \overset{\text{growth restriction}}{\leq} e^{-st} \cdot M \cdot e^{kt} = M \cdot e^{(k-s)t}$

$t \to \infty$: $e^{-st} \cdot f(t) \to 0$ if $k - s \leq 0$

• $t = 0$: $e^0 \cdot f(0) = f(0)$

• If f' is just piecewise continuous, the proof is similar. Then the interval of integration of f' must be broken up into parts s.t. f' is continuous in each such part.

• Use (1):

$$\mathcal{L}(f'') \overset{(1)}{=} s\cdot \mathcal{L}(f') - f'(0)$$

$$\overset{(1)}{=} s\cdot\left(s\cdot \mathcal{L}(f) - f(0)\right) - f'(0)$$

$$= s^2 \mathcal{L}(f) - s\cdot f(0) - f'(0)$$

Theorem: Laplace Transform of the Derivative $f^{(n)}$ of Any Order

Let $f, f', \ldots, f^{(n-1)}$ be continuous for all $t \geqq 0$ and satisfy the growth restriction[2]. Furthermore, let $f^{(n)}$ be piecewise continuous on every finite interval on the semi-axis $t \geqq 0$. Then the transform of $f^{(n)}$ satisfies

$$\mathcal{L}(f^{(n)}) = s^n\mathcal{L}(f) - s^{n-1}f(0) - s^{n-2}f'(0) - \ldots - f^{(n-1)}(0)).$$

(Proof by induction)

for all $i = 0, 1, \ldots, n-1$:

[2] $\exists\ M, k\ s.t.\ \forall t \geqq 0 :\ |f^{(i)}(t)| \leqq Me^{kt}$.

Example

Let $f(t) = t \sin \omega t$. Find $\mathcal{L}(f'')$ and $\mathcal{L}(f)$.

Example

Let $f(t) = \cos \omega t$ and $g(t) = \sin \omega t$. Find $\mathcal{L}(f)$ and $\mathcal{L}(g)$.

Theorem: Laplace Transform of Integral

Let $F(s)$ denote the transform of a function $f(t)$ which is piecewise continuous for $t \geqq 0$ and satisfies a growth restriction[3]. Then, for $s > 0$ and $t > 0$,

$$\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\} = \frac{1}{s}F(s), \text{ thus } \int_0^t f(\tau)d\tau = \mathcal{L}^{-1}\left\{\frac{1}{s}F(s)\right\}.$$

[3] $\exists\ M, k\ s.t.\ \forall t \geqq 0 :\ |f(t)| \leqq Me^{kt}$.

Example

Find the inverse of $\dfrac{1}{s(s^2+\omega^2)}$ and $\dfrac{1}{s^2(s^2+\omega^2)}$.

Differential Equations, Initial Value Problems

Let us now discuss how the Laplace transform method solves ODEs and initial value problems. We consider an initial value problem

$$y'' + ay' + by = r(t), \quad y(0) = K_0, \quad y'(0) = K_1,$$

where a and b are constant.
$r(t) \dots$
$y(t) \dots$

In Laplace's method we perform three steps:

Step 1. Setting up the subsidiary equation.

Step 2. Solution of the subsidiary equation by algebra.

Step 3. Inversion of Y to obtain $y = \mathcal{L}^{-1}(Y)$.

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Example

Solve

$$y'' - y = t, \quad y(0) = 1, \quad y'(0) = 1.$$

Comparison with the Usual Method

Example

Solve

$$y'' + y' + 9y = 0, \quad y(0) = 0.16, \quad y'(0) = 0.$$

Advantages of the Laplace Method

1. Solving a nonhomogeneous ODE does not require first solving the homogeneous ODE.
2. Initial values are automatically taken care of.
3. Complicated inputs $r(t)$ (right sides of linear ODEs) can be handled very efficiently, as we show later.

Shifted Data Problems

Example

Solve

$$y'' + y = 2t, \quad y\left(\frac{1}{4}\pi\right) = \frac{1}{2}\pi, \quad y'\left(\frac{1}{4}\pi\right) = 2 - \sqrt{2}.$$

Conclusion

$$F(s) := \mathcal{L}(f) := \int_0^\infty e^{-st} f(t)dt$$

What we have learned today

- Laplace Transform of the derivative of a function
- Laplace Transform of the integral
- Solving ODEs and Initial Value Problems using Laplace Transform
- Shifted data problems

Next Lecture

Chapter 6.3, 6.4 in Kreyszig

- Unit Step Function (Heaviside Function)
- Second Shifting Theorem (t-Shifting)
- Dirac's Delta Function

References

The material of this lecture was based on Chapter 6.2 of the book

Advanced Mathematical Engineering by Erwin Kreyszig (John Wiley & Sons, 10th edition, 2011)

and Chapter 6 in

Differential Equations Demystified by Steven G. Krantz (McGraw-Hill, 2005).

Moreover, we recommend the lecture notes by Morten Nome (in Norwegian), who taught the 2019 edition of this course. You can download Lecture 1 of Morten's lecture notes collection here:

https://www.math.ntnu.no/emner/TMA4125/2019v/notater/01-laplacetransform.pdf