Øving 11

Norges teknisk–naturvitenskapelige universitet
Institutt for matematiske fag

1 Required Exercises

1.1 The Superposition Principle

A PDE satisfies the **superposition principle** if, given any two solutions u, v and some $c \in \mathbb{R}$, the following conditions are satisfied:

- $u + v$ and $c \cdot u$ satisfy the equation.
- If initial or boundary conditions are given, then $u+v$ and $c \cdot u$ both satisfy the initial conditions.

1 For the following PDEs, check if the superposition holds.

a) $\frac{\partial^3 u}{\partial t^3} = x\frac{\partial u}{\partial x}$,

b) $\frac{\partial^2 u}{\partial t^2} = u\frac{\partial^2 u}{\partial x^2}$,

c) $\frac{\partial^2 u}{\partial t \partial x} = (t+x)\frac{\partial u}{\partial t}$, with $u(0,t) = 0$ and $\frac{\partial u}{\partial x}(1,t) = 0$,

d) $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$, with $u(0,t) = 0$ and $u(2,t) = 2$.

1.2 The Heat Equation

2 In this question we consider the heat equation with $c = 1$,

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \tag{1}$$

modelling the flow of heat through a thin, uniform rod of length L.

Consider first the case where we have two Dirichlet boundary conditions,

$$u(0,t) = 0, \text{ and } u(L,t) = 0. \tag{2}$$

a) Find the solution to (1) with boundary conditions (2) and with initial condition

$$u(x,0) = 7\sin\left(\frac{3\pi}{L}x\right) + 2\sin\left(\frac{5\pi}{L}x\right). \tag{3}$$

You may make use of the solution formula derived in lectures.

b) Find the solution to the inhomogeneous heat equation,

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 1, \tag{4}$$

with boundary conditions (2), and initial condition (3).
Hint: Consider a new function $v(x,t) = u(x,t) + \frac{1}{2}x(x-L)$, where u is a solution to (4). What initial and boundary conditions does it satisfy if u satisfies (2) and (3)?

c) Find the general solution to (1) now with Neumann boundary conditions

$$\frac{\partial u}{\partial x}(0,t) = 0, \text{ and } \frac{\partial u}{\partial x}(L,t) = 0,$$

via the separation of variables method, i.e. assuming $u(x,t) = F(x)G(t)$ for some functions F and G.
Your solution should be of the form

$$u(x,t) = \sum_{n\in\mathbb{N}} A_n g_n(k_n x) e^{-k_n t},$$

where g_n and k_n are unknowns for you to determine.
Given some initial data $u(x,0) = f(x)$, provide a formula to calculate A_n similar to the one from the Dirichlet BC problem.

2 Highly Recommended Exercises

2.1 The Wave Equation

3 Consider the one dimensional wave equation modelling a thing string of length L.

$$\frac{\partial^2 u}{\partial t^2} = c\frac{\partial^2 u}{\partial x^2}. \tag{5}$$

where $c > 0$ is some positive constant, with Neumann boundary conditions

$$\frac{\partial u}{\partial x}(0,t) = 0, \text{ and } \frac{\partial u}{\partial x}(L,t) = 0, \tag{6}$$

and initial condition

$$u(x,0) = f(x), \text{ and } u_t(x,0) = g(x), \tag{7}$$

for some given suitable functions f, g.

a) Using the separation of variables method, find the general solution formula for this problem.

b) Find the solution to equation (5) with $c = 4$ and initial data

$$u(x,0) = x^2, \text{ and } u_t(x,0) = \sin(x). \tag{8}$$