Norwegian University of Science and Technology
Department of Mathematical Sciences

TMA4130/35
Mathematics 4N/4D
Fall 2021

Exercise set 3

Mandatory problems

1 Let $f(x) = x^4 - x^3 + 4x^2 - x - 7$, for $0 = a \leq x \leq b = 2$.

a) Using the intermediate value theorem, show that there is a point $c \in [a, b]$, which satisfies the equation

$$f'(c) = \frac{f(b) - f(a)}{b - a}. \tag{1}$$

b) Determine with the help of the *bisection method* an interval $[s, t] \subseteq [0, 2]$, with $c \in [s, t]$ and $|t - s| \leq 1/16$.
Note: The uniqueness of the solution to the equation (1) in the interval $[a, b]$ does not need to be shown!

2 The function $f(x) = 1 - \sin(x)$, $x \in \mathbb{R}$ is given.

a) What are the fixed points of f?

b) Determine the largest possible interval around these fixed points on which f is contracting.

c) Calculate the first five approximations of the fixed point algorithm with the starting value $x_0 = \pi/4$.

d) Calculate with the same starting value as in c) five approximations for the fixed point by using the Newton method on the function $g(x) = x - f(x)$.

3 Using Newton's method, we want to approximate the number e. We are looking for a solution x to the equation $\ln x = 1$ in the interval $(2, 3)$.

a) Explain that there is a solution in $(2, 3)$.

b) Use the starting value $x_0 := 3$ to calculate two Newton iterations and estimate the error.

Additional exercises

These additional exercises are completely optional and should not be handed in. The student assistants will not grade these problems.

4 We want to apply fixed point iteration to the solution of the equation $\cos(x) = \frac{1}{2}\sin(x)$ using the mapping

$$g(x) = x + \cos(x) - \frac{1}{2}\sin(x)$$

with an initialisation $x_0 = 0$.

a) Show that the mapping g satisfies the conditions for the fixed point theorem on the interval $[a, b] = [0, \pi/2]$.

b) Provide an estimate of the accuracy of the outcome of the method after the 5^{th} iteration. How many iterations will be needed to obtain a result with an error smaller than 10^{-12}?

Hint: Use the a-priori error estimate in the fixed point theorem.

5 We consider the equation

$$f(x) = x^3 - x^2 - x + 1 = 0,$$

which has the two roots $r = -1$ and $r = +1$. In this exercise, we will discuss the usage of the Newton method in order to solve this equation. One can show that Newton's method for the solution of this equation converges for each initialisation $x_0 > 1$ to the root $r = 1$, and for each intitialisation $x_0 < -1$ to the root $r = -1$.

a) Write down the iteration scheme and perform the first three iterations with initialization $x_0 = 2$.

b) Based upon the theory developed in the lecture, what convergence order do you expect for the iterations starting with $x_0 = 2$ and $x_0 = -2$, respectively?

c) Use and modify the function `newton` in the Jupyter notes in order to verify the convergence orders numerically.

6 **a)** Compute 3 iterations by hand of Newton's method to approximate a solution to the system of equations

$$\begin{aligned} x^2 + y^2 &= 4 \\ xy &= 1 \end{aligned}$$

starting with $x_0 = 2$, $y_0 = 0$.

b) Find an approximation to the solution by using the function **Newton_system**. You can also use this to confirm that your hand calculations were correct.

c) Use **Newton_system** to solve the slightly perturbed system.

$$\begin{aligned} x^2 + y^2 &= 2 \\ xy &= 1 \end{aligned}$$

with the same initial values. How does the method behave now? Explain the results.