

Lecture 16

Numerical solutions of nonlinear equations

Introduction

Here, we will consider some numerical techniques for solving

- nonlinear scalar equations
(one equation, one unknown),
such as

$$x^3 + x^2 + 2x = 3,$$

- or system of equations, for example

$$\begin{aligned} x \cdot e^y &= 1 \\ -x^2 + y &= 1 \end{aligned}$$

Scalar equations

A scalar equation is given by

$$f(x) = 0,$$

with f being a continuous function defined on some interval $[a, b]$.

A solution r of the equation is called root of f .

Example : $f(v) = v^2 - 2$, $v \in \mathbb{R}$.

$f(x) = x^3 + x^2 - 3x - 5$
 on $[-2, 2]$

$\Rightarrow f$ has three real roots in $[-2, 2]$

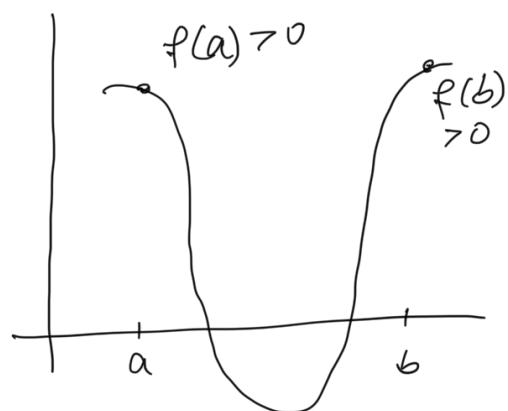
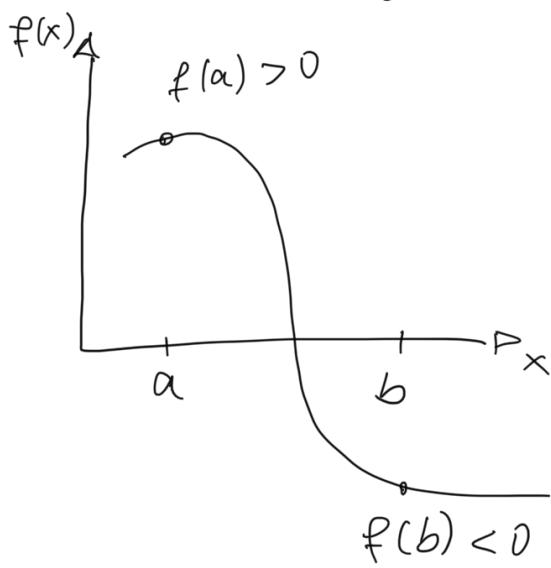
$$\text{Re-write } f : \quad f(x) = x^3 + x^2 - 3x - 3 \\ = (x+1)(x^2 - 3)$$

$$\Rightarrow \text{roots are } -1, \pm\sqrt{3}$$

Existence and Uniqueness of Solutions

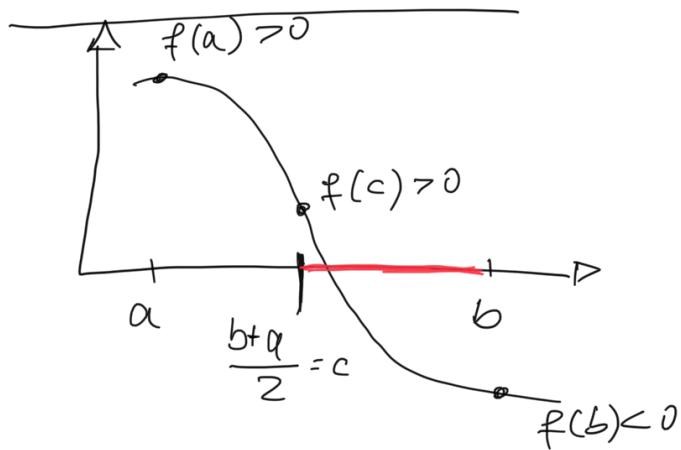
Theorem : • If $f \in C[a, b]$ with $f(a)$ and $f(b)$ of opposite sign, then there exists at least one $r \in (a, b)$ s.t. $f(r) = 0$.

- The solution is unique if $f \in C^1[a, b]$ and $f'(x) > 0$ or $f'(x) < 0$ for all $x \in (a, b)$. (f = the function is either strictly increasing or strictly decreasing).



\Rightarrow first condition is sufficient (not necessary)

Bisection Method



Steps : Given: The function f and the interval $[a, b]$ s.t.

$$f(a) \cdot f(b) < 0.$$

$$\cdot \text{ set } a_0 = a, b_0 = b$$

• For $k = 0, 1, 2, 3, \dots$

$$c_k = \frac{a_k + b_k}{2}$$

$$[a_{k+1}, b_{k+1}] = \begin{cases} [a_k, c_k] & \text{if } f(a_k) \cdot f(c_k) \leq 0 \\ [c_k, b_k] & \text{if } f(c_k) \cdot f(b_k) \leq 0 \end{cases}$$

The error satisfies: $|c_k - r| \leq \frac{b_k - a_k}{2}$
 ↑
 approximation
 of the root

The loop is terminated when $\frac{b_k - a_k}{2}$ is smaller than some user-specified tolerance, or if $f(c_k)$ is very close to zero

Fixed Point Iterations

- Given a scalar equation $f(x) = 0$ with root r .
- Re-write the equation in the fixed point form $x = g(x)$ s.t. the root r of f is a fixed point of g , that is, $r = g(r)$.

Fixed Point Iterations

- Given g and a starting value x_0
 - For $k = 0, 1, 2, 3, \dots$
- $$x_{k+1} = g(x_k).$$

Theory : let us apply the theorem above to the equation
 $f(x) = x - g(x) = 0$.

Then : • If $g \in [a, b]$ and
 $a < g(x) < b$ for all $x \in [a, b]$,
then g has at least one fixed point $r \in (a, b)$.

- If, in addition, $g' \in C^1[a, b]$ and
 $|g'(x)| < 1$ for all $x \in [a, b]$,
then the fixed point is unique
(and the iterations converge to the unique point).

the unique fixed point).

Note that the condition

$$\forall x \in [a, b] : a < g(x) < b$$

is equivalent to

$$\Leftrightarrow \boxed{\forall x \in [a, b] : g(x) \in (a, b)}$$

- $e_k = r - x_k$... error after k iterations
The iterations converge when $e_k \rightarrow 0$ as $k \rightarrow \infty$.

We are interested in conditions that ensure convergence:

For arbitrary k , we have:

$$x_{k+1} = g(x_k) \quad (\text{the iterations})$$

$$r = g(r) \quad (\text{the fixed point})$$

$$\Rightarrow |e_{k+1}| = |r - x_{k+1}| \\ = |g(r) - g(x_k)|$$

mean-value-theorem

$$\downarrow \\ = |g'(t_k)| \cdot |r - x_k|$$

for t_k between x_k and r

$$= |g'(s_k)| \cdot |e_k|$$

- $g([a,b]) \subset (a,b)$ guarantees that if $\underbrace{x_0 \in [a,b]}_{\Rightarrow g(x_0) \in (a,b)}$, then $x_k \in (a,b)$ for $k = 1, 2, 3, \dots$
 $= x_1$

- The condition $|g'(x)| \leq L < 1$ guarantees convergence towards the unique fixed point r , since

$$\frac{|e_{k+1}|}{|e_{k+1}|} \leq L \cdot |e_k|$$

$$|e_{k+1}| \leq L \cdot |e_k| \leq L |e_{k-1}| \cdot L \leq \dots$$

$$\Rightarrow |e_k| \leq L^k |e_0| \rightarrow 0$$

and $L^k \rightarrow 0$ as $k \rightarrow \infty$
 when $L < 1$.

- The fixed point is unique:
 Assume $r_1 \neq r_2$ are two fixed points.
 Then

$$\begin{aligned} |r_1 - r_2| &= |g(r_1) - g(r_2)| \\ &= |g'(s)| |r_1 - r_2| \\ &< 1 \cdot |r_1 - r_2| \end{aligned}$$

Thus $r_1 = r_2$ the following fixed point

This leads to the ~~Intermediate~~^{Intermediate} Value Theorem:

Fixed Point Theorem: If there is an interval $[a, b]$ s.t. $g \in C^1[a, b]$, $g([a, b]) \subset (a, b)$ and there exists a positive constant $L < 1$ s.t.

$|g'(x)| \leq L < 1$ for all $x \in [a, b]$, then

- g has a unique fixed point r in (a, b) .
- The fixed point iterations $x_{k+1} = g(x_k)$ converge towards r for all starting values $x_0 \in [a, b]$.

Conclusion: • The smaller the constant L , the faster the convergence

- If $|g'(r)| < 1$, then there will always be a neighborhood around r on which all the conditions are satisfied.

\Rightarrow The iterations will always converge if x_0 is sufficiently close to r .

- If $|g'(r)| > 1$, the fixed point

iterations will never converge towards r .

Example : $g(x) = \frac{x^3 + x^2 - 3}{3}$

$$\Rightarrow g'(x) = \frac{3x^2 + 2x}{3}$$

g has three fixed points :

$$r = \pm \sqrt{3}$$

$$\text{and } r = -1$$

$$g'(\sqrt{3}) = 3 + \frac{2}{3}\sqrt{3} \\ = 4.15$$

$$g'(-1) = \frac{3-2}{3} \\ = \frac{1}{3} < 1$$

$$g'(-\sqrt{3}) = 3 - \frac{2}{3}\sqrt{3} \\ = 1.85$$

assumption is the fixed point theorem are not fulfilled

assumption is fulfilled

Newton's Method

By the previous discussion, fast convergence can be achieved if $g'(r)$ is as small as possible, preferably $g'(r) = 0$.

- error in iteration k :

$$e_k = r - x_k \Rightarrow x_k = \underline{\underline{r - e_k}}$$

$$\begin{aligned}
 \rightarrow e_{k+1} &= r - x_{k+1} \\
 &= g(r) - g(\underline{x_k}) \\
 &= g(r) - g(r - e_k) \\
 \text{Taylor expansion} \\
 &\stackrel{\downarrow}{=} -\underline{g'(r)} \cdot e_k + \frac{1}{2} \underline{g''(\beta_k)} \cdot e_k^2
 \end{aligned}$$

If $g'(r) = 0$ and there exists

$$M: \frac{g''(x)}{2} \leq M \quad .$$

$$|e_{k+1}| \leq M |e_k|^2$$

\Rightarrow quadratic convergence.

Question: Gives the equation $f(x) = 0$
with an unknown solution r ($f(r) = 0$)
 Can we find a g with r as a
 fixed point, satisfying $g'(r) = 0$?

Idea: Let $\underline{g(x) := x - h(x) \cdot f(x)}$

for some function $h(x)$.

$$\begin{aligned}
 \text{Then we have: } g(r) &= r - h(r) \cdot \underbrace{f(r)}_{=0} \\
 &= r,
 \end{aligned}$$

and so r is a fixed point of g .

choose $h(x)$ s.t. $g'(r) = 0$, that is,

$$\begin{aligned}
 g'(x) &= 1 - h'(x) \cdot f(x) - h(x) \cdot f'(x) \\
 g'(r) &= 1 - h'(r) \cdot f(r) - h(r) \cdot f'(r)
 \end{aligned}$$

$$\overbrace{= \overset{'}{0}}^{\text{def}}$$

$$\Rightarrow h(r) \cdot f'(r) = 1$$

$$\Rightarrow h(x) = \frac{1}{f'(x)}$$

=====

$$\Rightarrow g(x) = x - \frac{f(x)}{f'(x)}$$

is the iteration step in Newton's method

Newton's Method

- Given : f and a starting value x_0

- For $k = 0, 1, 2, \dots$

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

Error Analysis : The method was constructed to give quadratic convergence, that,

$$|e_{k+1}| \leq M \cdot |e_k|^2,$$

where $e_k = r - x_k$ is the error after k iterations. But under which conditions is this true, and can we

say something about the size of the constant M ?

Taylor expansion

$$(1) \quad 0 = f(r) \stackrel{\downarrow}{=} f(x_k) + f'(x_k)(r - x_k) \\ + \frac{1}{2} \cdot f''(\xi_k)(r - x_k)^2$$

with ξ_k between r and x_k .

From the Newton's method, we get:

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

$$\Rightarrow \frac{f(x_k)}{f'(x_k)} = x_k - x_{k+1}$$

$$\Rightarrow f(x_k) = (x_k - x_{k+1}) \cdot f'(x_k)$$

$$\Rightarrow 0 = (x_k - x_{k+1}) \cdot f'(x_k) - f(x_k)$$

$$(2) \quad \Rightarrow 0 = f(x_k) + f'(x_k)(x_{k+1} - x_k)$$

Subtraction (2) from (1), we get:

$$\begin{aligned} & \cancel{f(x_k)} + f'(x_k)(r - \cancel{x_k}) + \frac{1}{2} f''(\xi_k)(r - x_k) \\ & - \cancel{f(x_k)} - f'(x_k)(x_{k+1} - \cancel{x_k}) \\ & = f'(x_k) \underbrace{(r - x_{k+1})}_{= e_{k+1}} + \frac{1}{2} f''(\xi_k) \underbrace{(r - x_k)}_{= e_k^2} \\ & = 0 \end{aligned}$$

$$\Rightarrow e_{k+1} = \frac{-\frac{1}{2} f''(\xi_k) \cdot e_k^2}{f'(x_k)}$$

\Rightarrow so, we obtain quadratic convergence if f is twice differentiable around r , $f'(x_k) \neq 0$ and x_0 is chosen sufficiently close to r .

Theorem : Convergence of Newton Iteration

Assume that the function f has a root r , and let $I_\delta = [r-\delta, r+\delta]$ for some δ . Assume further :

- $f \in C^2(I_\delta)$
- There is a positive constant M s.t.

$$\frac{f''(y)}{f'(x)} \leq 2M \quad \forall x, y \in I_\delta.$$

Then Newton's iterations converge quadratically,

$$|e_{k+1}| \leq M \cdot |e_k|^2$$

for all starting values satisfying

$$|x - x_0| \leq \min \left\{ \frac{1}{M}, \delta \right\}$$