Lecture 1: Laplace Transform. Linearity. First Shifting Theorem (s-Shifting)

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Literature

Erwin Kreyszig. *Advanced Mathematical Engineering*. John Wiley & Sons, 10th edition, 2011

Wikipage: `https://wiki.math.ntnu.no/tma4130/2020h/start`

Solving linear ODEs and related initial value problems

$$\boxed{\text{Initial Value Problem (IVP) ODE}} \xrightarrow[\text{transform}]{\text{Laplace}} \boxed{\text{Algebraic Problem (AP) algeb. equation}} \Longrightarrow \boxed{\text{Solving AP by Algebra}} \Longrightarrow \boxed{\text{Solution of the IVP}}$$

(subsidiary equation
- no derivative)

$my''(t) + cy'(t) + k$

t ... time

$f(t)$... incoming signal

What is a transform?

The idea of a transform is that it turns a given function into another function. We are already acquainted with several transforms:

1. The derivative D takes a differentiable function f (defined on some interval (a, b)) and assigns to it a new function $Df := f'$.
2. The integral I takes a continuous function f (defined on some interval $[a, b]$) and assigns to it a new function

$$If(t) := \int_a^t f(x)dx.$$

3. The multiplication operator M_ϕ, which multiplies any given function f on the interval $[a, b]$ by a fixed function ϕ on $[a, b]$, is a transform:

$$M_\phi f(t) := \phi(t) \cdot f(t).$$

We are particularly interested in transforms that are linear. A transform T is *linear* if for some functions f, g and constants $\alpha, \beta \in \mathbb{R}$

$$\boxed{T[\alpha f + \beta g] = \alpha T(f) + \beta T(g).}$$

In particular (taking $\alpha = \beta = 1$),

$$\boxed{T[f + g] = T(f) + T(g),}$$

and (taking $\beta = 0$)

$$\boxed{T[\alpha f] = \alpha T(f).}$$

Definition: Laplace transform

Given a function $f(t)$ ($f : \mathbb{R}_+ \to \mathbb{R}$), its **Laplace transform** is defined as

input

output

$$F(s) := \mathscr{L}(f) := \int_0^{\infty} e^{-st} f(t)dt.$$

The given function $f(t)$ is called the **inverse transform** of $F(s)$ and is denoted by $\mathscr{L}^{-1}(F)$. Note that the above integral is an **improper integral**, which is evaluated according to the rule

$$\boxed{\int_0^{\infty} e^{-st} f(t)dt := \lim_{T\to\infty} \int_0^{T} e^{-st} f(t)dt.}$$

Remark

Original functions are denoted by lowercase letters and their transforms by the same letters in capital, so that $F(s)$ denotes the transform of $f(t)$, and $Y(s)$ denotes the transform of $y(t)$, and so on.

Laplace Transform

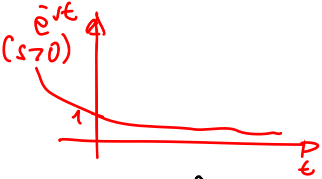
Figure: Pierre-Simon, marquis de Laplace, 23 March 1749 – 5 March 1827, French astronomer and mathematician.

Example

Let $f(t) = 1$ when $t \geqq 0$. Find $F(s)$.

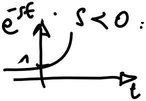
$$\mathcal{L}(f) = \int_0^\infty e^{-st} \cdot 1 = \left[-\frac{1}{s} e^{-st} \right]_0^\infty = 0 - \left(-\frac{1}{s} \cdot \underbrace{e^{-s \cdot 0}}_{=1} \right)$$

$$= \underline{\underline{\frac{1}{s}}} \quad (s > 0)$$



· $s = 0$: $\int_0^\infty \underbrace{e^0}_{=1} dt = \infty \Rightarrow \mathcal{L}(f)$ does not exist.

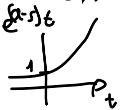
· $s < 0$: $\int_0^\infty e^{-st} dt = \infty \Rightarrow \mathcal{L}(f)$ does not exist.



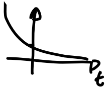
Example

Let $f(t) = e^{at}$ when $t \geqq 0$, where a is a constant. Find $F(s)$.

$$\mathcal{L}(f) = \int_0^\infty e^{-st} \cdot e^{at}\, dt = \int_0^\infty e^{(a-s)t}\, dt = \left[\frac{1}{a-s} \cdot e^{(a-s)t} \right]_0^\infty$$



- $a - s > 0 \Rightarrow \mathcal{L}(f)$ does not exist.



- $a - s < 0 \Rightarrow \mathcal{L}(f) = 0 - \frac{1}{a-s} \underbrace{e^{(a-s)\cdot 0}}_{=1} = \underline{\underline{\frac{1}{s-a}}}$

- $a = s \Rightarrow \mathcal{L}(f) = \int_0^\infty e^{0t}\, dt = \int_0^\infty 1\, dt = \infty$

  $\Rightarrow \mathcal{L}(f)$ does not exist.

$a = 0 \Rightarrow f(t) = e^{0t} = 1 \Rightarrow \mathcal{L}(f) = \frac{1}{s} \; (s > 0)$

in accordance with the preceding example.

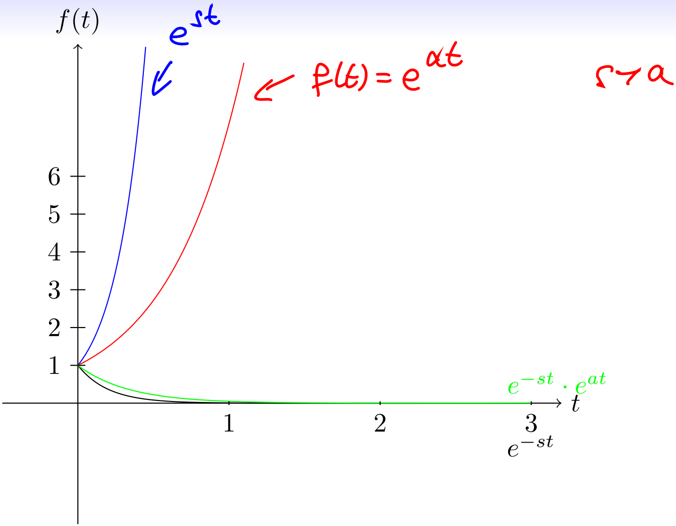


Figure: Visual verification that the Laplace transform of e^{at} exists only if $s > a$: Functions e^{st} with $s = 5$ and $f(t) = e^{at}$ with $a = 2$.

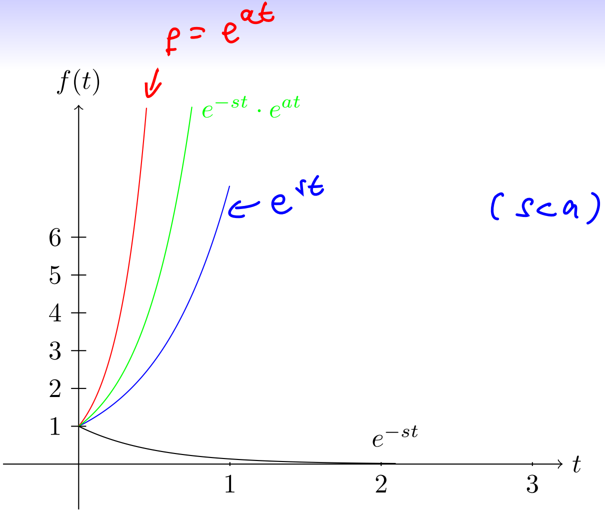


Figure: Visual verification that the Laplace transform of e^{at} does not exists if $s < a$: Functions e^{st} with $s = 2$ and $f(t) = e^{at}$ with $a = 5$.

A Non-Working Example

Example

Does the Laplace transform of $f(t) = e^{t^2}$ exist?

$$\mathcal{L}(f) = \int_0^\infty e^{-st} \cdot e^{t^2}\, dt$$

$$= \int_0^\infty e^{t(t-s)}\, dt$$

(t → ∞, at one point, t will be bigger than s)

$= \infty \Rightarrow \mathcal{L}(f)$ does not exist

e^{t^2} increases much faster than e^{-st} decreases.

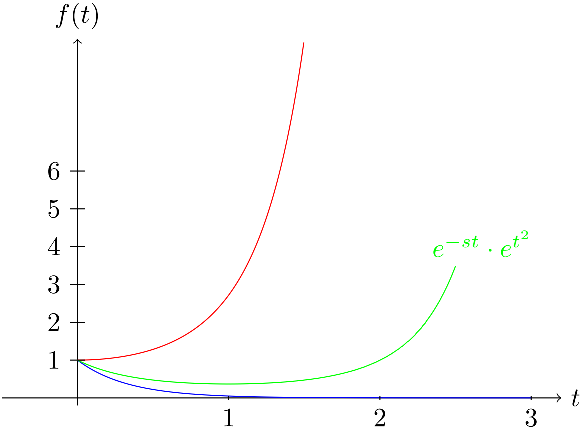


Figure: Functions $f(t) = e^{t^2}$ and e^{-st} for $s = 2$.

Theorem: Linearity of the Laplace Transform

The Laplace transform is a linear operation; that is, for any functions f and g whose transforms exist and any constants a and b the transform of $af(t) + bg(t)$ exists, and

$$\boxed{\mathscr{L}\{af(t) + bg(t)\} = a\mathscr{L}\{f(t)\} + b\mathscr{L}\{g(t)\}.}$$

Proof: $\mathcal{L}\{af(t) + bg(t)\} = \int_0^\infty e^{-st}(af(t) + bg(t))\,dt$

$= \int_0^\infty e^{-st} af(t)\,dt + \int_0^\infty e^{-st} b\,g(t)\,dt$

$= a \cdot \int_0^\infty e^{-st} f(t)\,dt + b\int_0^\infty e^{-st} g(t)\,dt$

$= a \cdot \mathcal{L}(f) + b \cdot \mathcal{L}(g).$ ∎

Example

Find the transforms of $\cosh at$ and $\sinh at$. $(a \in \mathbb{R},\ a \neq 0)$

$$\cosh at = \frac{1}{2} \cdot \left(e^{at} + e^{-at}\right)$$

$$\mathcal{L}(\cosh at) = \frac{1}{2}\mathcal{L}(e^{at}) + \frac{1}{2}\mathcal{L}(e^{-at})$$

$$= \frac{1}{2} \cdot \underbrace{\frac{1}{s-a}}_{s>a} + \frac{1}{2} \cdot \underbrace{\frac{1}{s+a}}_{s>-a} \qquad \text{if } s > |a|$$

$$= \frac{1}{2} \cdot \frac{s+a + s-a}{(s-a)(s+a)}$$

$$= \frac{1}{2} \cdot \frac{2s}{s^2-a^2}$$

$$= \frac{s}{s^2-a^2}$$

$$\sinh at = \frac{1}{2}\left(e^{at} - e^{-at}\right)$$

$$\mathcal{L}(\sinh at) = \frac{1}{2}\mathcal{L}(e^{at}) - \frac{1}{2}\mathcal{L}(e^{-at})$$

$$= \frac{1}{2}\cdot\frac{1}{s-a} - \frac{1}{2}\cdot\frac{1}{s+a} \qquad (s > |a|)$$

$$= \frac{1}{2}\cdot\frac{s+a-(s-a)}{(s-a)(s+a)}$$

$$= \frac{1}{2}\cdot\frac{2a}{s^2-a^2}$$

$$= \frac{a}{s^2-a^2}$$

Example

Derive the formulas

$$\mathscr{L}(\cos \omega t) = \frac{s}{s^2+\omega^2}$$

and

$$\mathscr{L}(\sin \omega t) = \frac{w}{s^2+\omega^2}.$$

Integration by parts:

$\int g' f = g \cdot f - \int g f'$

$$\mathcal{L}(\cos wt) = \int_0^\infty \underbrace{e^{-st}}_{g'} \cdot \underbrace{\cos wt}_{f} \, dt$$

$$= \Big[\underbrace{-\tfrac{1}{s} e^{-st}}_{g} \cdot \underbrace{\cos wt}_{f}\Big]_0^\infty - \int_0^\infty \left(-\tfrac{1}{s} e^{-st}\right)(-w \sin wt)\,dt$$

$$= 0 - \left(-\tfrac{1}{s}\, 1 \cdot 1\right) - \frac{w}{s} \cdot \int e^{-st} \sin wt \, dt$$

$$= \frac{1}{s} - \frac{w}{s} \cdot \mathcal{L}(\sin wt)$$

$$\mathcal{L}(\sin wt) = \int_0^\infty \underbrace{e^{-st}}_{g'} \cdot \underbrace{\sin wt}_{f} \, dt$$

$$= \Big[-\tfrac{1}{s} e^{-st} \sin wt\Big]_0^\infty + \int_0^\infty \frac{w}{s} e^{-st} \cos wt \, dt$$

$$= 0 - \left(-\tfrac{1}{s} \cdot 1 \cdot 0\right) + \frac{w}{s} \cdot \mathcal{L}(\cos wt)$$

$$= \frac{w}{s}\, \mathcal{L}(\cos wt)$$

$\mathcal{L}(\sin wt) =: \mathcal{L}_s$

$\mathcal{L}(\cos wt) =: \mathcal{L}_c$

$\Rightarrow \mathcal{L}_c = \frac{1}{s} - \frac{w}{s} \mathcal{L}_s$

$\mathcal{L}_s = \frac{w}{s} \mathcal{L}_c$

$\Rightarrow \mathcal{L}_c = \frac{1}{s} - \frac{w}{s} \cdot \frac{w}{s} \mathcal{L}_c$

$\frac{1}{s} = \mathcal{L}_c + \frac{w^2}{s^2} \mathcal{L}_c$

$= \frac{s^2 + w^2}{s^2} \mathcal{L}_c$

$\Rightarrow \mathcal{L}_c = \frac{1}{s} \cdot \frac{s^2}{s^2+w^2} = \frac{s}{s^2+w^2}$

$\mathcal{L}_s = \frac{w}{s} \cdot \mathcal{L}_c = \frac{w}{s} \cdot \frac{s}{s^2+w^2} = \frac{w}{s^2+w^2}$

Some Functions $f(t)$ and Their Laplace Transforms $\mathcal{L}(f)$

	$f(t)$	$\mathcal{L}(f)$		$f(t)$	$\mathcal{L}(f)$
1	1	$1/s$	7	$\cos \omega t$	$\dfrac{s}{s^2 + \omega^2}$
2	t	$1/s^2$	8	$\sin \omega t$	$\dfrac{\omega}{s^2 + \omega^2}$
3	t^2	$2!/s^3$	9	$\cosh at$	$\dfrac{s}{s^2 - a^2}$
4	t^n $n=3$ $(n = 0, 1, \cdots)$	$\dfrac{n!}{s^{n+1}}$ $\dfrac{3!}{s^4}$	10	$\sinh at$	$\dfrac{a}{s^2 - a^2}$
5	t^a (a positive)	$\dfrac{\Gamma(a + 1)}{s^{a+1}}$	11	$e^{at} \cos \omega t$	$\dfrac{s - a}{(s - a)^2 + \omega^2}$
6	e^{at}	$\dfrac{1}{s - a}$	12	$e^{at} \sin \omega t$	$\dfrac{\omega}{(s - a)^2 + \omega^2}$

Example

Find the Laplace transform of the function

$$f(t) = 5t^3 - 2e^t.$$

$$\mathcal{L}(f) = 5 \cdot \mathcal{L}(t^3) - 2 \cdot \mathcal{L}(e^t)$$

$$= 5 \cdot \frac{3!}{s^4} - 2 \cdot \frac{1}{s-1}$$

Theorem: First Shifting Theorem, s-Shifting

If $f(t)$ has the transform $F(s)$ (where $s > k$ for some k), then $\boxed{e^{at} f(t)}$ has the transform $F(s-a)$ (where $s - a > k$). In formulas,

$$\boxed{\mathscr{L}\{e^{at} f(t)\} = F(s-a),}$$

or, if we take the inverse on both sides,

$$e^{at} f(t) = \mathscr{L}^{-1}\{F(s-a)\}.$$

<u>Proof:</u>

$$\mathcal{L}\{e^{at} f(t)\} = \int_0^\infty e^{-st} \cdot e^{at} f(t)\, dt$$

$$= \int_0^\infty e^{-(s-a)t} f(t)\, dt$$

$$= F(s-a)$$

If $F(s)$ exists (i.e., is finite) for $s > k$, then $F(s-a)$ also exists for $s - a > k$. ■

Example

Find the inverse of the transform

$$\mathscr{L}(f) = \frac{3s - 137}{s^2 + 2s + 401}.$$

We know: $\mathcal{L}\{e^{at}\cos\omega t\} = \dfrac{s-a}{(s-a)^2+\omega^2}$

$\mathcal{L}\{e^{at}\sin\omega t\} = \dfrac{\omega}{(s-a)^2+\omega^2}$

$$f = \mathcal{L}^{-1}\left\{\frac{3(s+1)-140}{(s+1)^2+400}\right\}$$

$\mathcal{L}^{-1}$ linear

$$\overset{\downarrow}{=} 3\cdot\mathcal{L}^{-1}\left\{\frac{\underbrace{s+1}_{=-a}}{(s+1)^2+\underbrace{20}_{=\omega}{}^2}\right\} - 7\cdot\mathcal{L}^{-1}\left\{\frac{\overbrace{20}^{=\omega}}{(s+1)^2+20^2}\right\}$$

$$= \underline{\underline{3\cdot e^{-t}\cdot\cos 20t - 7e^{-t}\cdot\sin 20t}}.$$

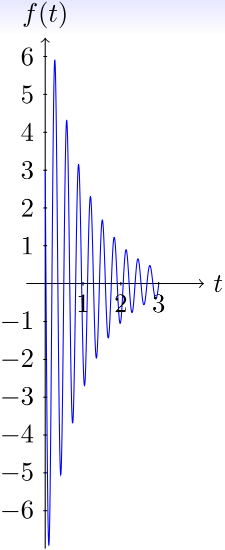


Figure: Function $f(t) = e^{-t} \cdot (3\cos 20t - 7\sin 20t)$.

Existence and Uniqueness of Laplace Transforms

1. $f(t)$ should satisfy the **growth restriction**

$$\boxed{\exists\ M, k\ s.t.\ \forall t \geqq 0 :\ |f(t)| \leqq Me^{kt}.} \qquad (2)$$

2. $f(t)$ should be **piecewise** continuous on a finite interval $a \leqq t \leqq b$ where f is defined. That is, this interval can be divided into finitely many subintervals in each of which f is continuous and has **finite** limits as t approaches either endpoint of such a subinterval from the interior.

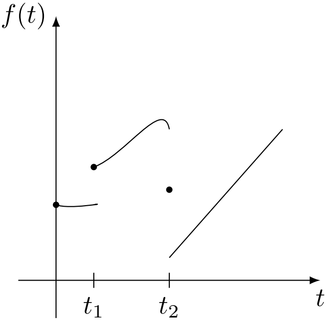


Figure: A piecewise continuous function $f(t)$

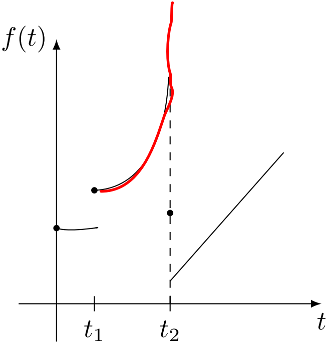


Figure: $f(t)$ is **not** piecewise continuous.

Theorem: Existence Theorem for Laplace Transforms

If $f(t)$ is defined and piecewise continuous on every finite interval on the semi-axis $t \geqq 0$ and satisfies (2) for all $t \geqq 0$ and some constants M and k, then the Laplace transform $\mathscr{L}(f)$ exists for all $s > k$.

Proof: Since $f(t)$ is piecewise continuous, $e^{-st} \cdot f(t)$ is integrable over any finite number on the t-axis.

$$|\mathcal{L}(f)| = \left| \int_0^\infty \underbrace{e^{-st}}_{\geq 0} f(t)\, dt \right| \leq \int_0^\infty |f(t)| \cdot e^{-st}\, dt$$

$$\overset{(2)}{\leq} \int_0^\infty M \cdot e^{kt} \cdot e^{-st}\, dt = \frac{M}{s-k} \quad (\text{if } s > k)$$

$\Rightarrow \mathcal{L}(f)$ exists for all $s > k$ ∎

Uniqueness

If the Laplace transform of a given function exists, it is uniquely determined. Conversely, it can be shown that if two functions (both defined on the positive real axis) have the same transform, these functions cannot differ over an interval of positive length, although they may differ at isolated points. Hence we may say that the inverse of a given transform is essentially unique. In particular, if two continuous functions have the same transform, they are completely identical.

References

The material of this lecture was based on Chapter 6.1 of the book

Advanced Mathematical Engineering by Erwin Kreyszig (John Wiley & Sons, 10th edition, 2011)

and Chapter 6 in

Differential Equations Demystified by Steven G. Krantz (McGraw-Hill, 2005).

Moreover, we recommend the lecture notes by Morten Nome (in Norwegian), who taught the 2019 edition of this course. You can download Lecture 1 of Morten's lecture notes collection here:

`https://www.math.ntnu.no/emner/TMA4125/2019v/notater/01-laplacetransform.pdf`