Lecture 10: Fourier Transform. Discrete and Fast Fourier Transforms

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Complex Form of the Fourier Integral

As already seen for the Laplace transform, an integral transform is a mapping that outputs a function from another function using an integral expression over the dependent variable of the input function. The main purpose of integral transforms is in the solution of ODEs and PDEs. To define the **Fourier transform** as the second most important integral transform in this field, we use the Fourier integral representation of functions and simply rescale the involved functions $C(w)$.

$$
\begin{aligned}
f(x) &= \int_{-\infty}^{\infty} \frac{1}{2\pi} \cdot \int_{-\infty}^{\infty} f(v) \cdot \mathrm{e}^{-\mathrm{i}vw}\, dv \cdot \mathrm{e}^{\mathrm{i}wx}\, dw \\
&= \frac{1}{\sqrt{2\pi}} \cdot \int_{-\infty}^{\infty} \underbrace{\left(\frac{1}{\sqrt{2\pi}} \cdot \int_{-\infty}^{\infty} f(v) \cdot \mathrm{e}^{-\mathrm{i}vw}\, dv \right)}_{=:\hat{f}(w)} \cdot \mathrm{e}^{\mathrm{i}wx}\, dw
\end{aligned}
$$

Complex Form of the Fourier Integral

This leads to the following definition.

Definition

The expression

$$\mathscr{F}(f)(w) = \hat{f}(w) := \frac{1}{\sqrt{2\pi}} \cdot \int_{-\infty}^{\infty} f(x) \cdot \mathrm{e}^{-\mathrm{i}wx}\, dx$$

is called the (complex) **Fourier transform** of f.
On the other hand, the expression

$$\mathscr{F}^{-1}(\hat{f})(x) = f(x) = \frac{1}{\sqrt{2\pi}} \cdot \int_{-\infty}^{\infty} \hat{f}(w) \cdot \mathrm{e}^{\mathrm{i}wx}\, dw$$

is called the inverse Fourier transform of $\hat{f}$.

Existence of the Fourier Transform

Theorem

If $f(x)$ is absolutely integrable on the x-axis and piecewise continuous on every finite interval, then the Fourier transform $\hat{f}(w)$ of $f(x)$ exists.

Example

Find the Fourier transform of $f(x) = 1$ if $|x| < 1$ and $f(x) = 0$ otherwise.

Example

Find the Fourier transform of $f(x) = \exp(-ax)$ if $x > 0$ and $f(x) = 0$ if $x < 0$, where $a > 0$.

Linearity of the Fourier Transform

Linearity of the Fourier Transform

Fourier Transform of the Derivative of $f(x)$

Fourier Transform of the Derivative of $f(x)$

Example

Find the Fourier transform of $f(x) = x \cdot \exp(-x^2)$.

Convolution

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