Matematikk 4N – 1. september

Derivasjon og integrasjon av Laplacetransformerte

[mål: endå flere muligheter for å beregne Laplacetransformerte eller inverse transformerte]

Anta at funksjonen $f(t)$ har $F(s)$ som Laplacetransformerte

$$\Rightarrow \quad F'(s) = \frac{d}{ds}\int_0^\infty e^{-st} f(t)\, dt$$

$$= \int_0^\infty \frac{d}{ds} e^{-st} f(t)\, dt$$

$$= \int_0^\infty (-t)\, e^{-st} f(t)\, dt = -\mathcal{L}(t f(t))$$

Dvs at:

$$\mathcal{L}(t f(t)) = -F'(s)$$

$$\mathcal{L}^{-1}(F'(s)) = -t f(t) = -t\,\mathcal{L}^{-1}(F(s)).$$

F.eks: Vet at $f(t) = \cos 2t$ har den Laplacetransformerte

$$F(s) = \frac{s}{s^2+4}$$

$$\Rightarrow \mathcal{L}(t\cdot\cos 2t) = -F'(s)$$

$$= -\frac{s^2+4-2s^2}{(s^2+4)^2} = \frac{s^2-4}{(s^2+4)^2}$$

Feks: Vil beregne den inverse transformerte til

$$F(s) = \frac{s}{(s^2-9)^2}.$$

$\rightsquigarrow$ Har at $F(s) = G'(s)$ for $G(s) = -\frac{1}{2}\cdot\frac{1}{s^2-9}$

Derfor er: $\mathcal{L}^{-1}(F(s)) = \mathcal{L}^{-1}(G'(s))$

$$= -t\cdot\mathcal{L}^{-1}(G(s))$$

Men $\mathcal{L}^{-1}(G(s)) = -\frac{1}{2}\cdot\frac{1}{3}\sinh 3t = -\frac{1}{6}\sinh 3t$

Dermed: $\mathcal{L}^{-1}(F(s)) = -t\cdot\left(-\frac{1}{6}\sinh 3t\right) = \frac{t}{6}\sinh 3t.$

Nå antar vi at $F(s) = \mathcal{L}(f(t))$. Så er

$$\int_s^\infty F(\sigma)\,d\sigma = \int_s^\infty\int_0^\infty e^{-\sigma t} f(t)\,dt\,d\sigma$$

$$= \int_0^\infty\int_s^\infty e^{-\sigma t} f(t)\,d\sigma\,dt$$

$$= \int_0^\infty \left(-\frac{1}{t}\right)e^{-\sigma t} f(t)\Big|_{\sigma=s}^{\infty}\,dt$$

$$= \int_0^\infty \frac{1}{t} e^{-st} f(t)\,dt = \mathcal{L}\left(\frac{f(t)}{t}\right)$$

Dvs: $$\mathcal{L}\left(\frac{f(t)}{t}\right) = \int_s^\infty F(\sigma)\,d\sigma$$

og $$\mathcal{L}^{-1}\left(\int_s^\infty F(\sigma)\,d\sigma\right) = \frac{\mathcal{L}^{-1}(F(s))}{t}$$

Feks.: Hvis $f(t) = \boxed{\frac{e^t - 1}{t}}$, så er:

$$f(t) = \frac{g(t)}{t}$$ og derfor:

$$\mathcal{L}(f) = \int_s^\infty \mathcal{L}(g)(\sigma)\,d\sigma$$

Her: $\mathcal{L}(g) = \mathcal{L}(e^t - 1) = \frac{1}{s-1} - \frac{1}{s}$ og

$$\mathcal{L}(f) = \int_s^\infty \frac{1}{\sigma - 1} - \frac{1}{\sigma}\,d\sigma$$

$$= \left(\ln(\sigma - 1) - \ln\sigma\right)\Big|_s^\infty$$

$$= \ln\left(\frac{\sigma - 1}{\sigma}\right)\Big|_s^\infty = \ln\left(1 - \frac{1}{\sigma}\right)\Big|_s^\infty$$

$$= \ln(1) - \ln\left(1 - \frac{1}{s}\right) = \boxed{-\ln\left(1 - \frac{1}{s}\right)}$$

$$\rightsquigarrow \mathcal{L}\left(\frac{e^t - 1}{t}\right) = -\ln\left(1 - \frac{1}{s}\right)$$

System av ODEer

Idé: som før: • transformer systemet
• løs for de transformerte
• transformer tilbake

F.eks.:

$$y_1' = 2y_1 + y_2 + e^{2t} \qquad y_1(0) = 0$$

$$y_2' = y_1 + 2y_2 \qquad y_2(0) = 1.$$

transformer systemet:

$$sY_1 - \underbrace{y_1(0)}_{=0} = 2Y_1 + Y_2 + \frac{1}{s-2}$$

$$sY_2 - \underbrace{y_2(0)}_{=1} = Y_1 + 2Y_2$$

dvs:

$$s \cdot Y_1 = 2Y_1 + Y_2 + \frac{1}{s-2}$$

$$s Y_2 - 1 = Y_1 + 2Y_2$$

$$\rightsquigarrow \quad (s-2) \cdot Y_1 - Y_2 = \frac{1}{s-2}$$

$$-Y_1 + (s-2) \cdot Y_2 = 1$$

eller:

$$\underbrace{\begin{bmatrix} s-2 & -1 \\ -1 & s-2 \end{bmatrix}}_{A} \cdot \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} 1/s-2 \\ 1 \end{bmatrix}$$

$$\rightsquigarrow \quad A^{-1} = \frac{1}{\underbrace{(s-2)^2 - 1}_{\det A}} \cdot \begin{bmatrix} s-2 & +1 \\ +1 & s-2 \end{bmatrix}$$

Og

$$\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \frac{1}{(s-2)^2-1} \begin{bmatrix} s-2 & +1 \\ +1 & s-2 \end{bmatrix} \cdot \begin{bmatrix} 1/s-2 \\ 1 \end{bmatrix}$$

$$= \frac{1}{s^2-4s+3} \cdot \begin{bmatrix} 2 \\ \frac{1}{s-2} + s - 2 \end{bmatrix}$$

$$\rightsquigarrow \quad Y_1 = \frac{2}{s^2-4s+3}$$

$$Y_2 = \frac{1}{s^2-4s+3} \cdot \left(\frac{1}{s-2} + s - 2 \right)$$

Delbrøksoppsp: $(s^2-4s+3) = (s-3) \cdot (s-1)$

og: $$\frac{1}{s^2-4s+3} = \frac{1}{2} \cdot \frac{1}{s-3} - \frac{1}{2} \cdot \frac{1}{s-1}$$

$$\rightsquigarrow \quad Y_1 = \frac{2}{s^2-4s+3} = \frac{1}{s-3} - \frac{1}{s-1}$$

$$\boxed{y_1(t) = e^{3t} - e^{t}}$$

Og: $$Y_2 = \frac{1}{2} \cdot \left(\frac{1}{s-3} - \frac{1}{s-1}\right) \cdot \left(\frac{1}{s-2} + s - 2\right)$$

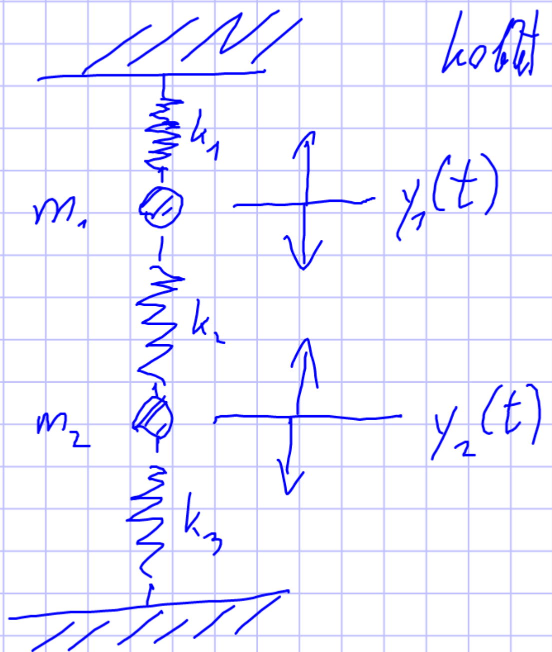
$$= \dots = \frac{1}{s-3} - \frac{1}{s-2} + \frac{1}{s-1}$$

delbrøk...

Og: $$\boxed{y_2(t) = e^{3t} - e^{2t} + e^{t}}$$

F.eks.: system av andre ordens ligninger

koblet masse-fjær system



$$\rightsquigarrow \quad m_1 y_1'' = -k_1 y_1 + k_2(y_2 - y_1)$$
$$m_2 y_2'' = -k_2(y_2 - y_1) - k_3 y_2$$

F.eks.: $m_1 = m_2 = 1$

$k_1 = k_3 = 1, \; k_2 = 4$

Og: $y_1(0) = 0, \quad y_1'(0) = 1$

$y_2(0) = 0, \quad y_2'(0) = -1$

System: $$y_1'' = -5y_1 + 4y_2$$
$$y_2'' = 4y_1 - 5y_2$$

Laplacetransformasjon gir:

$$s^2 Y_1 - s\underbrace{y_1(0)}_{=0} - \underbrace{y_1'(0)}_{=+1} = -5Y_1 + 4Y_2$$
$$s^2 Y_2 - s\underbrace{y_2(0)}_{=0} - \underbrace{y_2'(0)}_{=-1} = 4Y_1 - 5Y_2$$

$$\begin{cases} s^2 Y_1 - 1 = -5Y_1 + 4Y_2 \\ s^2 Y_2 + 1 = 4Y_1 - 5Y_2 \end{cases}$$

$$\begin{cases} (s^2+5)Y_1 - 4Y_2 = 1 & / \cdot 4 \\ -4Y_1 + (s^2+5)Y_2 = -1 & / \cdot (s^2+5) \end{cases}$$

$$\left.\begin{matrix} 4(s^2+5)Y_1 - 16Y_2 = 4 \\ -4(s^2+5)Y_1 + (s^2+5)^2 Y_2 = -s^2 - 5 \end{matrix}\right\} +$$

$$\Big(\underbrace{(s^2+5)^2 - 16}\Big) Y_2 = -s^2 - 1$$

$$s^4 + 10s^2 + 9 = (s^2+9)\cdot(s^2+1)$$

[løsningene til $s^4 + 10s^2 + 9 = 0$ er: $s^2 = -5 \pm \sqrt{25-9} = \begin{cases} -9 \\ -1 \end{cases}$

$$\rightsquigarrow \quad (s^2+9)\cdot\cancel{(s^2+1)}\,Y_2 = -\cancel{(s^2+1)}$$

$$Y_2 = -\frac{1}{s^2+9} \rightsquigarrow y_2(t) = -\frac{1}{3}\sin 3t$$

Og: Y_1: har at $-4Y_1 + (s^2+5)Y_2 = -1$

$$\rightsquigarrow$$

$$= -\frac{1}{s^2+9}$$

$$\rightsquigarrow \quad Y_1 = \frac{1}{s^2+9}$$

$$\rightsquigarrow \quad y_1(t) = \frac{1}{3}\sin 3t$$

Får løsningen
$$y_1(t) = 1/3 \sin 3t$$
$$y_2(t) = -1/3 \sin 3t$$