Norwegian University of Science and Technology Department of Mathematical Sciences	TMA4130 Mathematics 4N Fall 2020 **Exercise set 5**

Mandatory problems

1 Compute the following integrals and show your work in detail:

a) $\displaystyle\int_0^\infty \frac{\cos(xw) + w\sin(xw)}{1+w^2}dw$ (This exercise can be found in Kreyszig, p. 517, but note that there is a typo there.) Hint: Calculate the representation of the function

$$f(x) = \begin{cases} 0 & (x<0) \\ \pi\exp(-x) & (\text{else}) \end{cases}$$

by a Fourier integral.

b) $\displaystyle\int_0^\infty \frac{1-\cos(\pi w)}{w}\sin(xw)dw$. Hint: Calculate the representation of the function

$$f(x) = \begin{cases} \pi/2 & (0<x<\pi) \\ 0 & (\text{else}) \end{cases}$$

by a Fourier integral.

2 For some positive constant $\alpha > 0$, define the function

$$f(x) = \begin{cases} 1 & (|x|<\alpha) \\ 0 & (\text{else}) \end{cases}$$

and show that its Fourier transform is

$$\hat{f}(w) = \sqrt{\frac{2}{\pi}}\frac{\sin(\alpha w)}{w}.$$

3 Compute the Fourier transform of the following functions:

a) $f(x) = \begin{cases} 0 & (x \le 0) \\ \exp(-x) & (x>0) \end{cases}$

b) $f(x) = \begin{cases} 1-x^2 & (-1 \le x \le 1) \\ 0 & (|x|>1) \end{cases}$

c) $f(x) = \begin{cases} T+x & (-T \le x < 0) \\ T-x & (0 \le x \le T) \\ 0 & (|x|>T) \end{cases}$

4 Compute the convolution

$$(f * f)(x) = \int_{-\infty}^{\infty} f(x-p) f(p) dp$$

for the function $f(p) = \frac{1}{p^2+\lambda^2}$ for a given $\lambda > 0$.

Additional exercises

These additional exercises are completely optional and should not be handed in. The student assistants will not grade these problems.

5 Compute the Fourier transform of

$$f(x) = \begin{cases} \sin(x) & (-\pi < x < \pi) \\ 0 & (\text{else}) \end{cases}$$

and calculate the integral

$$\int_{-\infty}^{\infty} \frac{\sin(\pi w)\sin(\pi w/2)}{1-w^2} dw.$$

6 Let $f(x) = \exp(-x^2)$ and $g(x) = x\exp(-x^2)$. Show that

$$f * g = -\frac{i}{4}\int_{-\infty}^{\infty} w \exp(-\frac{w^2}{2}) \exp(iwx) dw.$$

7 Find the Fourier transform of $f(x) = x^2 \exp(-x^2)$. (Hint: It might be useful to differentiate $\exp(-x^2)$ twice.)