TMA4130 Matematikk 4N Solutions: 7th Week

1.

(i) Notice that the function $v(t,x) = u(t,x) - (U_1 + (U_2 - U_1)x/L)$ satisfies

$$\frac{\partial v}{\partial t} = \frac{\partial u}{\partial t}, \qquad \frac{\partial^2 v}{\partial x^2} = \frac{\partial^2 u}{\partial x^2},$$

so if u solves the heat equation with the boundary conditions $u(t,0) = U_1$, $u(t,L) = U_2$, then v satisfies the heat equation

$$\frac{\partial v}{\partial t} = c^2 \frac{\partial^2 v}{\partial x^2}$$

with the boundary conditions

$$v(t,0) = u(t,0) - U_1 = 0, \qquad v(t,L) = u(t,L) - U_1 - (U_2 - U_1)L/L = 0.$$

Therefore we can write v as

$$v(t,x) = \sum_{n>0} B_n \sin\left(\frac{n\pi}{L}x\right) e^{-(cn\pi/L)^2 t},$$

with

$$B_n = \frac{2}{L}\int_0^L (u(0,x) + (U_1 + (U_2 - U_1)x/L)) \sin\left(\frac{n\pi}{L}x\right) \mathrm{d}x,$$

and hence

$$u(t,x) = U_1 + (U_2 - U_1)\frac{x}{L} + \sum_{n>0} B_n \sin\left(\frac{n\pi}{L}x\right) e^{-(cn\pi/L)^2 t}.$$

Therefore in the $t \to \infty$ limit, the temperature profile is linear:

$$\lim_{t\to\infty} u(t,x) = U_1 + (U_2 - U_1)\frac{x}{L}.$$

(ii) This is Example 4 of cap. 12 sec. 6.

2. Taking the hint and setting $v(t,x) = u(t,x) + Hx(x-\pi)/(2c^2)$ we find that

$$\frac{\partial v}{\partial t} = \frac{\partial u}{\partial t}, \qquad \frac{\partial^2 v}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + H(x-\pi) + Hx\right) = \frac{\partial^2 u}{\partial x^2} + H/c^2.$$

Therefore if u solves

$$\frac{\partial v}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} + H,$$

then v solves

$$\frac{\partial v}{\partial t} = c^2 \frac{\partial^2 v}{\partial x^2}.$$

The boundary conditions are

$$v(t,0) = u(t,0) = 0, \qquad v(t,\pi) = u(t,\pi) = 0.$$

Solving the equation for v we find:

$$v(t,x) = \sum_{n>0} B_n \sin(nx) e^{-(cn)^2 t}$$

with

$$\begin{aligned} B_n &= \frac{2}{\pi}\int_0^\pi (u(0,x) + Hx(x-\pi)) \sin(nx)\, \mathrm{d}x \\ &= \frac{2}{\pi}\int_0^\pi u(0,x)\sin(nx)\, \mathrm{d}x + \frac{H}{n}\left((-1)^{n+1}(\pi^2 - \pi) + \frac{2}{n^2}((-1)^n - 1)\right). \end{aligned}$$

Finally we express the solution in terms of u:

$$u(t,x) = \sum_{n>0} B_n \sin(nx) e^{-(cn)^2 t} - Hx(x-\pi).$$

3. We apply separation of variables $u(x,y) = F(x)G(y)$. The equation becomes:

$$G\frac{\mathrm{d}^2F}{\mathrm{d}x^2} = -F\frac{\mathrm{d}^2G}{\mathrm{d}y^2}$$
$$\frac{1}{F}\frac{\mathrm{d}^2F}{\mathrm{d}x^2} = -\frac{1}{G}\frac{\mathrm{d}^2G}{\mathrm{d}y^2}.$$

These give us the ODEs

$$\frac{\mathrm{d}^2F}{\mathrm{d}x^2} = -kF, \qquad \frac{\mathrm{d}^2G}{\mathrm{d}y^2} = kG.$$

These have the solutions, respectively,

$$F(x) = A^{**}e^{\sqrt{-k}x} + B^{**}e^{-\sqrt{-k}x}$$
$$G(y) = C^{**}e^{\sqrt{k}y} + D^{**}e^{-\sqrt{k}y}.$$

We apply the top and bottom insulation boundary conditions that

$$\frac{\partial u}{\partial y}(x,0) = 0, \qquad \frac{\partial u}{\partial y}(x,a) = 0, \qquad \text{for } x \in (0,a).$$

To apply these conditions we differentiate u in y, which is

$$\frac{\partial u}{\partial y} = F\frac{\mathrm{d}G}{\mathrm{d}y}.$$

We are compelled by the boundary conditions to conclude that

$$\frac{\mathrm{d}G}{\mathrm{d}y}(0) = \frac{\mathrm{d}G}{\mathrm{d}y}(a) = 0.$$

These in turn restrict k to some negative $-p^2$. Writing

$$G(y) = C^* \cos(py) + D^* \sin(py),$$

so that

$$\frac{\mathrm{d}G}{\mathrm{d}y} = -C^* p \sin(py) + D^* p \cos(py)$$

we see that the insultation condition at $y = 0$ implies $D^* = 0$, and the insulation condition at $y = a$ implies the quantization of p into only taking values $p = n\pi/a$ for $n = 0, 1, \ldots$. Therefore we can write

$$G_n(y) = C_n^* \cos\left(\frac{n\pi}{a}y\right),$$

with the constant C_n^* to be determined by the other boundary conditions.

Taking this information to solve the equation for F we find that

$$F_n(x) = A_n^{**}e^{n\pi x/a} + B_n^{**}e^{-n\pi x/a} = A_n^* \cosh\left(\frac{n\pi}{a}x\right) + B_n^* \sinh\left(\frac{n\pi}{a}x\right).$$

(This is just expressing F_n in another basis — or, take $A_n^* = (A_n^{**} + B_n^{**})/2$ and $B_n^* = (A_n^{**} - B_n^{**})/2$.)

Since $u(0,y) = 0$, we find that $F_n(0) = 0$, so $A_n^* = 0$. That leaves us with

$$u(x,y) = \sum_{n>0} C_n \cos\left(\frac{n\pi}{a}y\right) \sinh\left(\frac{n\pi}{a}x\right),$$

where $C_n = C_n^* B_n^*$. Notice that this means $C_0 = 0$.

We stlil have one boundary condition to use. Setting $x = a$, we find

$$f(y) = u(a, y) = \sum_{n>0} C_n \sinh(n\pi) \cos\left(\frac{n\pi}{a} y\right).$$

This is a cosine series from which we can extract an expression for C_n assuming convergence of the series foregoing by

$$C_n = \frac{2}{a \sinh(n\pi)} \int_0^a f(y) \cos\left(\frac{n\pi}{a} y\right) \, \mathrm{d}y$$

for $n > 0$.

4. This question is about evaluating Fourier transforms.

(i) We use the fact that the Fouier transform of $e^{-|x|}$ is $2\pi^{-1/2}(1 + x^2)^{-1}$.

$$C(p) = \int_{\mathbb{R}} \frac{1}{1 + x^2} e^{-ipx} \, \mathrm{d}x = \pi e^{-|p|}$$

See also Sheet 5, Problems 2(iii) and 3(ii).

(ii) Again we use the fact that the Fourier transform of $\mathbb{1}_{[-1,1]}$ is $2(2\pi)^{-1} \sin(x)/x$:

$$C(p) = \int_{\mathbb{R}} \frac{\sin(x)}{x} e^{-ipx} \, dx = \pi \mathbb{1}_{[-1,1]}(p).$$

See also Sheet 5, Problems 2(ii) and 3(ii).

(iii) Finally integration against the delta function is evaluation at 0 (see Lecture III).

$$C(p) = \int_{\mathbb{R}} \delta(x) e^{-ipx} \, dx = e^{ip(0)} = 1.$$