# TMA4130 Matematikk 4N Problem Sheet: 7th Week

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**1.**

(i) (Problem 8 of cap. 12 sec. 6) If the ends $x = 0$ and $x = L$ of a one-dimensional bar of uniform linear density, thermal conductivity, and specific heat were in heat baths of temperatures $u(t, 0) = U_1$ and $u(t, L) = U_2$, $U_2 > U_1 > 0$, what is the asymptotic profile $u(t, x)$ as $t \to \infty$?

(ii) (Problem 11 of cap. 12 sec. 6) Show that for a completely insulated bar with Neumann boundary conditions

$$\frac{\partial u}{\partial x}(t,0) = \frac{\partial u}{\partial x}(t,L) = 0,$$

and initial condition $u(0, x) = f(x)$ (typo in book here), the separation of variables solution is given by

$$u(t,x) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) e^{-(cn\pi/L)^2 t}.$$

**2.** (Problem 16 of cap. 12 sec. 6) Solve the equation

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} + H$$

for a bar modelled by the interval $[0, \pi]$, with the ends kept at $u(t, 0) = u(t, \pi) = 0$. *(Hint: set $u(t, x) = v(t, x) - Hx(x - \pi)/(2c^2)$.)*

**3.** (Problem 23 of cap. 12 sec. 6) Find the steady-state temperature of a square two dimensional plate of uniform planar density, thermal conductivity, and specific heat. Assume this plate is parameterised by having corners at $\{(0, 0), (0, a), (a, a), (a, 0)\}$, and the upper and lower sides are perfectly insulated whilst the left side is kept at $u(0, y) = 0$ and the right side is kept at a fixed temperature profile $u(a, y) = f(y)$.

**4.** (Problems 3 and 7 of cap. 12 sec. 7) Using Eq. (6) of cap. 12 sec. 7, find solutions to the heat equation on $\mathbb{R}$ with the following initial conditions (hints in book):

(i) $u(0, x) = (1 + x^2)^{-1}$,
(ii) $u(0, x) = \sin(x)/x$,
(iii) $u(0, x) = \delta(x)$ (see 3(ii) of Sheet 2, or Fourier inversion formula).

That is, find the real and imaginary parts of the coefficients $C(p)$ such that

$$u(t,x) = \int_{\mathbb{R}} C(p) e^{ipx} e^{-(cp)^2 t}\, \mathrm{d}p$$

for the foregoing initial conditions.