TMA4130 Matematikk 4N Solutions: 6th Week

1. Using separation of variables $u(t, x) = F(x)G(t)$,

$$\frac{\partial^2 u}{\partial t^2} = F\frac{\mathrm{d}^2 G}{\mathrm{d}t^2},$$

and

$$-c^2\frac{\partial^4 u}{\partial x^4} = -c^2\frac{\mathrm{d}^4 F}{\mathrm{d}x^4}G.$$

Equating them and putting each to one side,

$$\frac{-1}{c^2 G}\frac{\mathrm{d}^2 G}{\mathrm{d}t^2} = \frac{1}{F}\frac{\mathrm{d}^4 F}{\mathrm{d}x^4} = k,$$

for some constant k.

We need k to be positive otherwise G increases without bound either forward or backwards in time.

This gives us $k = \alpha^2$. For convenience because F satisfies a fourth-order equation we can freely set $\alpha = \beta^2$.

The expressions are seen clearly to general solutions simply by verifying that they satisfy the equations, and have the appropriate number of degrees of freedom.

2.

(i) From $u(0, t) = 0$ we gather that

$$0 = F(0)G(t) = (A + C)G(t) \Rightarrow A = -C.$$

Differentiating F we have

$$0 = \frac{\partial^2 u}{\partial x^2}(0, t) = (-A\beta^2 + C\beta^2)G(t) \Rightarrow A = C.$$

These conditions imply that $A = C = -C = 0$.

From the other boundary,

$$0 = u(L, t) = (B(\sin(\beta L) + D\sinh(\beta L))G(t).$$

and

$$0 = \frac{\partial^2 u}{\partial x^2}(L, t) = (-B\beta^2\sin(\beta L) + D\beta^2\sinh(\beta L))G(t).$$

Adding these up we find

$$0 = D\sinh(\beta L)G(t).$$

A non-trivial solution requires $D = 0$, and therefore also

$$B\sin(\beta L) = 0.$$

Therefore we find that $\beta = n\pi/L$, and

$$F_n(x) = B_n \sin\left(\frac{n\pi}{L}x\right),$$

and

$$G_n(t) = a_n \cos\left(\frac{c\pi^2 n^2}{L^2}t\right) + b_n \cos\left(\frac{c\pi^2 n^2}{L^2}t\right).$$

(ii) For a clamped beam, the slopes are zero at the ends:

$$0 = u(0, t) = u(0, L) = \frac{\partial u}{\partial x}(0, t) = \frac{\partial u}{\partial x}(L, t).$$

This means that again, on the one end,

$$0 = u(0, t) = (A + C)G(t) \Rightarrow A = -C,$$

which is unchanged from (i), and secondly,

$$0 = \frac{\partial u}{\partial x}(0,t) = \beta(B+D)G(t) \Rightarrow B = -D.$$

From the other end,

$$\begin{aligned} u(L,t) =& G(t)(A\cos(\beta L) + B\sin(\beta L) + C\cosh(\beta L) + D\sinh(\beta L)) \\ =& G(t)(A(\cos(\beta L) - \cosh(\beta L)) + B(\sin(\beta L) - \sinh(\beta L))). \end{aligned}$$

and

$$\begin{aligned} \frac{\partial u}{\partial x}(L,t) =& \beta G(t)(-A\sin(\beta L) + B\cos(\beta L) + C\sinh(\beta L) + D\cosh(\beta L)) \\ =& \beta G(t)(-A(\sin(\beta L) + \sinh(\beta L)) + B(\cos(\beta L) - \cosh(\beta L))). \end{aligned}$$

By taking $A = \cos(\beta L) - \cosh(\beta L)$, $B = \sin(\beta L) + \sinh(\beta L)$, the equation for $(\partial u/\partial t)(L,t)$ would be set to nought.

Now in the equation for $u(t,L)$

$$\begin{aligned} & A(\cos(\beta L) - \cosh(\beta L)) + B(\sin(\beta L) - \sinh(\beta L)) \\ =& \cos^2(\beta L) - 2\cos(\beta L)\cosh(\beta L) + \cosh^2(\beta L) + \sin^2(\beta L) - \sinh^2(\beta L). \end{aligned}$$

Using the identities

$$\cos^2(\beta L) + \sin^2(\beta L) = 1$$
$$\cosh^2(\beta L) - \sinh^2(\beta L) = 1,$$

clearly the condition

$$\cos(\beta L)\cosh(\beta L) = 1$$

allows us to satisfy $u(t,L) = 0$.

3. We write the equations as

$$A(x,y)\frac{\partial^2 u}{\partial x^2} + 2B(x,y)\frac{\partial^2 u}{\partial x \partial y} + C(x,y)\frac{\partial^2 u}{\partial y^2} = 0,$$

and test for the sign of the discriminant $B^2 - AC$.

(i)

$$\frac{\partial^2 u}{\partial x^2} - 16\frac{\partial^2 u}{\partial y^2} = 0,$$

This is a wave equation with $c = 4$ (or $1/4$ depending on which is interpreted as the temporal variable). Therefore it is hyperbolic and has solution of the form

$$u(t,x) = \Phi(x+ct) + \Psi(x-ct).$$

(ii)

$$\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = 0,$$

This is parabolic as $A = 1$, $B = 1$, $C = 1$ and $B^2 - AC = 0$.

This means one of the new variables is $\xi = x$, and the other new variable can be found using the characteristic equation:

$$\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 - 2\frac{\mathrm{d}y}{\mathrm{d}x} + 1 = 0$$

Clearly this can be solved by setting $y = mx + \eta$, where m satisfies $m^2 - 2m + 1 = 0$. Therefore the characteristics are $\eta = y - x$. The normal form is

$$\frac{\partial^2 u}{\partial \xi^2} = 0,$$

and d'Alembert's solution is

$$u(x, y) = \xi \Phi(\eta) + \Psi(\eta) = x\Phi(x - y) + \Psi(x - y).$$

(iii)

$$\frac{\partial^2 u}{\partial x^2} - 2\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = 0,$$

This is parabolic as $A = 1$, $B = -1$, $C = 1$ and $B^2 - AC = 0$.

This means one of the new variables is $\xi = x$, and the other new variable can be found using the characteristic equation:

$$\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + 2\frac{\mathrm{d}y}{\mathrm{d}x} + 1 = 0$$

Just as above, this yields characteristics of the form $\eta = y + x$. The normal form is

$$\frac{\partial^2 u}{\partial \xi^2} = 0,$$

and d'Alembert's solution is

$$u(x, y) = \xi \Phi(\eta) + \Psi(\eta) = x\Phi(x + y) + \Psi(x + y).$$

(iv)

$$x\frac{\partial^2 u}{\partial x^2} - y\frac{\partial^2 u}{\partial x \partial y} = 0.$$

This is (essentially) hyperbolic as $A = x$, $B = -y/2$, $C = 0$, and $B^2 - AC = y^2/4 \geq 0$ except at $y = 0$.

We solve the characteristic equation to get the characteristics:

$$x\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + y\frac{\mathrm{d}y}{\mathrm{d}x} = 0.$$

This tells us that either $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ which gives us the characteristics $\xi = y$ or

$$x\frac{\mathrm{d}y}{\mathrm{d}x} + y = 0.$$

The ODE is solved by $y = \eta/x$, for some constant η, so $\eta = xy$ is another characteristic. Since the coefficients are not constant, we write out the transformation for clarity:

$$\begin{aligned} x\frac{\partial^2 u}{\partial x^2} =& x\left(\frac{\partial^2 u}{\partial \xi^2}\left(\frac{\partial \xi}{\partial x}\right)^2 + 2\frac{\partial^2 u}{\partial \xi \partial \eta}\frac{\partial \eta}{\partial x}\frac{\partial \xi}{\partial x} + \frac{\partial^2 u}{\partial \eta^2}\left(\frac{\partial \eta}{\partial x}\right)^2\right) \\ -y\frac{\partial^2 u}{\partial x \partial y} =& -y\left(\frac{\partial^2 u}{\partial \xi^2}\frac{\partial \xi}{\partial x}\frac{\partial \xi}{\partial y} + \frac{\partial^2 u}{\partial \xi \partial \eta}\left(\frac{\partial \eta}{\partial x}\frac{\partial \xi}{\partial y} + \frac{\partial \eta}{\partial y}\frac{\partial \xi}{\partial x}\right) + \frac{\partial^2 u}{\partial \eta^2}\frac{\partial \eta}{\partial x}\frac{\partial \eta}{\partial y}\right). \end{aligned}$$

Using the characteristics/change-of-coordinates $\xi = y$ and $\eta = xy$, we can add up the foregoing and find that

$$0 = x\frac{\partial^2 u}{\partial x^2} - y\frac{\partial^2 u}{\partial x \partial y} = -y^2\frac{\partial^2 u}{\partial \xi \partial \eta} = -\xi^2\frac{\partial^2 u}{\partial \xi \partial \eta}.$$

We can integrate the above by integrating in η first, and crucially leave out a constant:

$$\xi^2\frac{\partial u}{\partial \xi} = \phi(\xi) + C.$$

Dividing through by ξ^2, we can integrate again in ξ and incorporate $-C$ into Ψ to get:

$$u(x,y) = \Phi(\xi) + \frac{1}{\xi}\Psi(\eta) = \Phi(y) + \frac{1}{y}\Psi(xy),$$

where $\Phi(\xi) = \int \phi(\xi)/\xi \, \mathrm{d}\xi$.

(iv*) It was pointed out to me by a student that a less ad-hoc/ trick dependent method was possible by simply observing that as $\xi = y$, we can end up with

$$\begin{aligned}\frac{\partial^2 u}{\partial x \partial y} =& \frac{\partial^2 u}{\partial x \partial \xi} \\ =& \frac{\partial}{\partial \xi}\left(\frac{\partial u}{\partial \eta}\frac{\partial_\eta}{\partial x}\right) \\ =& \frac{\partial^2 u}{\partial \eta \partial \xi}\xi + \frac{\partial u}{\partial \eta}.\end{aligned}$$

Therefore,

$$0 = x\frac{\partial^2 u}{\partial x^2} - y\frac{\partial^2 u}{\partial x \partial y} = -\xi^2\frac{\partial^2 u}{\partial \eta \partial \xi} + \xi\frac{\partial u}{\partial \eta}.$$

This is much easier to integrate and gets you the correct answer too.

4. To show that the curves are characteristics, we simply verify that they satisfy the characteristic equation.

(i) It is elliptic if $x > 0$ and hyperbolic if $x < 0$ by testing the sign of $B^2 - AC$ as in Q3 above. Using separation of variables — $u(x,y) = F(x)G(y)$ —

$$xF\frac{\mathrm{d}^2 G}{\mathrm{d}y^2} + G\frac{\mathrm{d}^2 F}{\mathrm{d}x^2} = 0.$$

Therefore

$$\frac{1}{xF}\frac{\mathrm{d}^2 F}{\mathrm{d}x^2} = \frac{1}{G}\frac{\mathrm{d}^2 G}{\mathrm{d}y^2},$$

and each side is equal to the same constant k. In particular,

$$\frac{\mathrm{d}^2 F}{\mathrm{d}x^2} - kxF = 0.$$

By a change of variable, $\xi = k^{1/3}x$, and setting $\tilde{F}(k^{1/3}x) = F(x)$, we find

$$\frac{\mathrm{d}^2 \tilde{F}}{\mathrm{d}\xi^2} - \xi\tilde{F} = 0,$$

which is the Airy Equation.

Writing $y = C \pm \frac{2}{3}(-x)^{3/2}$, we simply evaluate:

$$(y')^2 + x = ((-x)^{1/2})^2 + x = 0,$$

as sought.

(ii) This equation is also elliptic if $x > 0$ and hyperbolic if $x < 0$ by testing the sign of $B^2 - AC$ as in Q3 above. Writing $y = C \pm \frac{1}{2}(-x)^{1/2}$, we find that

$$x(y')^2 + 1 = x((-x)^{-1/2})^2 + 1 = 0,$$

as sought.