TMA4130 Matematikk 4N Problem Sheet: 6th Week

published: 23/09/2019, scripts due: 07/10/2019, 50% for approval

1. Problem 15 of cap. 12 sec. 3:
The beam equation for a uniform beam without external static or dynamic loading is given by

$$\frac{\partial^2 u}{\partial t^2} = -c^2 \frac{\partial^4 u}{\partial x^4}.$$

Use separation of variables $u(t,x) = F(x)G(t)$ to show that

$$\begin{aligned}\frac{1}{F}\frac{\mathrm{d}^4 F}{\mathrm{d}x^4} =& -\frac{1}{c^2 G}\frac{\mathrm{d}^2 G}{\mathrm{d}t} = \beta^4,\\ F(x) =& A\cos(\beta x) + B\sin(\beta x) + C\cosh(\beta x) + D\sinh(\beta x)\\ G(t) =& a\cos(c\beta^2 t) + b\sin(c\beta^2 t).\end{aligned}$$

for some constants $\beta, A, B, C, D, a, b \in \mathbb{R}$.

2. Problems 16, 19 of cap 12. sec. 3:

(i) Find solutions $u_n(t,x) = F_n(x)G_n(t)$ to the beam equation in 1. above for a beam of length L corresponding to zero initial velocity and satisfying the boundary conditions:

$$u(0,t) = u(L,t) = 0, \qquad \frac{\partial^2 u}{\partial x^2}(0,t) = \frac{\partial^2 u}{\partial x^2}(L,t) = 0.$$

(ii) What are reasonable boundary conditions for the clamped beam (see Fig. 293 (B) on pg. 552)? Show that F in 1. above satisfies clamped beam boundary conditions if

$$\cosh(\beta L)\cos(\beta L) = 1.$$

3. Problems 10, 11, 12, and 15 of cap. 12 sec. 4:
Classify the following equations into types, reduce to a normal form, and use d'Alembert's method/method of characteristics to write down generalized solutions to the following equations:

(i)

$$\frac{\partial^2 u}{\partial x^2} - 16\frac{\partial^2 u}{\partial y^2} = 0,$$

(ii)

$$\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = 0,$$

(iii)

$$\frac{\partial^2 u}{\partial x^2} - 2\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = 0,$$

(iv)

$$x\frac{\partial^2 u}{\partial x^2} - y\frac{\partial^2 u}{\partial y^2} = 0.$$

4.

(i) Problem 20 of cap. 12 sec. 4: Show that the TRICOMI EQUATION,

$$x\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial x^2} = 0,$$

is of mixed-type. Use separation-of-variables $u(x, y) = F(x)G(y)$ to obtain from the Tricomi equation the AIRY EQUATION:

$$F'' - xF = 0.$$

Show that the solutions of the characteristic equation (i.e., the CHARACTERISTICS) are

$$y \pm \frac{2}{3}(-x)^{2/3} = C$$

for any constant C.

(ii) Show that the KELDYSH EQUATION,

$$x\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$

is of mixed-type. Show that the solutions of characteristic equation are

$$y \pm \frac{1}{2}(-x)^{1/2} = C$$

Many important problems in fluid mechanics and differential geometry can be reduced to questions about the Tricomi equation, in particular, problems in transonic flow (the region straddling subsonic and supersonic flow) and isometric embedding problems. Likewise, many problems in shock reflection-deflection in gas dynamics are reducible to questions concerning the Keldysh equation.