Supplementary Challenging Problems for 5th Week

(not for exercise classes)

Define the Fourier transform in the normalized way:

$$\hat{f}(\xi) = \int_{\mathbb{R}} f(x)e^{-2\pi i x\xi}\,\mathrm{d}x.$$

We say a function is "compactly supported" if outside some interval of finite length, it is identically 0. A function is smooth if it is continuously differentiable to any given order.

1.

(i) Let $f : \mathbb{R} \to \mathbb{R}$ be a bounded, continuous function of compact support on $\mathbb{R}$. Show that $F(x) = \sum_{n\in\mathbb{Z}} f(x+n)$ is a bounded periodic function, and find its period.

(ii) For a fixed integer k show that $(\mathscr{F}F)(k) = \hat{f}(k)$ — bearing in mind (a) that F is periodic and f is not so, and (b) that $e^{2\pi i m} = 1$ for any integer m.

(iii) By expressing F as its associated Fourier series, show that for $f : \mathbb{R} \to \mathbb{R}$ integrable and continuous,

$$\sum_{k\in\mathbb{Z}} \hat{f}(k) = \sum_{n\in\mathbb{Z}} f(n).$$

(iv) Use the formula in (iii) with $f(x) = e^{-x^2\lambda}$ to show that

$$\int_{\mathbb{R}} e^{-x^2}\,\mathrm{d}x = \sqrt{\pi}.$$

This startling formula is Poisson's Summation Formula, which is the starting point of much number-theoretical investigations. It can also be used to count the number of integer points in a set $E \subseteq \mathbb{R}^d$ (using the d-dimensional analogue) by setting $f = \mathbb{1}_E$ (or some slightly smoothened version thereof).

2. Let $T_x : f(x) \mapsto xf(x)$ be the multiplication operator on compactly supported smooth functions. Let $T_d : f(x) \mapsto (2\pi i)^{-1}\mathrm{d}f(x)/\mathrm{d}x$ be a differentiation operator on compactly supported smooth functions.

(i) Show that for any compactly supported smooth function f,

$$[T_x, T_d]f := T_x(T_d f) - T_d(T_x f) = (2\pi i)^{-1} f.$$

(ii) Use the Cauchy-Schwarz inequality for two compactly supported smooth functions f and g:

$$\left|\int f(x)g(x)\,\mathrm{d}x\right|^2 \le \int |f(x)|^2\,\mathrm{d}x \cdot \int |g(y)|^2\,\mathrm{d}y.$$

to show that for a compactly supported smooth function ψ, and any $x_0, \xi_0 \in \mathbb{R}$,

$$\int |\psi(x)|^2\,\mathrm{d}x \le C\left(\int |x-x_0|^2|\psi(x)|^2\,\mathrm{d}x\right)^{1/2}\left(\int |\xi-\xi_0|^2|(\hat{\psi})(\xi)|^2\,\mathrm{d}x\right)^{1/2}.$$

(Hint: Write $\int |f|^2\,\mathrm{d}x$ as $C\int |Tx(T_d f) - T_d(T_x f)|^2\,\mathrm{d}x$, expand the derivatives, and then apply the Cauchy-Schwarz inequality. Only after that, use Parseval's inequality to write $\int |\mathrm{d}f/\mathrm{d}x|^2\,\mathrm{d}x$ as $C\int |\xi-\xi_0|^2|\hat{f}^2\,\mathrm{d}\xi$.)

Interpreting $|\psi|^2$ as a probability distribution (so that $\int |\psi(x)|^2\,\mathrm{d}x = 1$), T_d as the momentum operator and T_x as the position operator, this is Heisenberg's uncertainty principle. It says that the spread in position (x), and the spread in its Fourier transform (the spread in momentum) cannot simultaneously be small.