TMA4130 Matematikk 4N Solutions: 5th Week

1.

(i) Since the Fourier transform of the even extension of $g(x) = e^{-x}\mathbb{1}_{[0,\infty)}$ is $(2\pi)^{-1/2}(1+\xi^2)^{-1}$ it holds by the inversion formula that

$$e^{-|x|} = \frac{2}{\pi}\int_0^\infty \frac{\cos(xw)}{1+w^2}\,\mathrm{d}w,$$

likewise the odd extension of g has the Fourier transform $(2\pi)^{-1/2}\xi(1+\xi^2)^{-1}$. Therefore

$$e^{-x}\mathbb{1}_{[0,\infty)} - e^{x}\mathbb{1}_{(-\infty,0]} = \frac{2}{\pi}\int_0^\infty \frac{w\sin(xw)}{1+w^2}\,\mathrm{d}w.$$

Therefore it remains only to invoke linearity and add up the two to get

$$\frac{1}{\pi}\int_0^\infty \frac{\cos(xw)+w\sin(xw)}{1+w^2}\,\mathrm{d}w = e^{-x}\mathbb{1}_{[0,\infty)} - 1/2\mathbb{1}_{\{0\}}.$$

(ii) By considering the odd function $f(x) = \pi/2\cdot\mathrm{sgn}(x)$, and integral

$$\frac{2}{\pi}\int_0^\infty f(x)\sin(wy)\,\mathrm{d}y = \frac{1-\cos(\pi w)}{w},$$

we can repeat the calculations in (i). However, we provide another, perhaps more tedious, but simpler derivation:

First notice that by Example 2 of cap. 11 sec. 7 in the book,

$$\frac{2}{\pi}\int_0^\infty \frac{\sin(xw)}{w}\,\mathrm{d}w = \frac{2}{\pi}\int_0^\infty \frac{\sin(xw)}{xw}\,\mathrm{d}(xw) = \begin{cases} 1 & x>0 \\ 0 & x=0 \\ -1 & x<0 \end{cases}.$$

Similarly, by inspecting the point $x=0$, for $x\geq 0$,

$$\begin{aligned}\frac{2}{\pi}\int_0^\infty \frac{\cos(\pi w)}{w}\sin(xw)\,\mathrm{d}w &= \frac{2}{\pi}\int_0^\infty \frac{\cos(\pi y/x)}{y}\sin(y)\,\mathrm{d}y \\ &= \begin{cases} 1 & 0\leq \pi/x<1 \\ 1/2 & \pi/x=1 \\ 0 & \pi/x>1 \end{cases} \\ &= \begin{cases} 1 & \pi<x\leq\infty \\ 1/2 & x=\pi \\ 0 & 0\leq x<\pi \end{cases},\end{aligned}$$

and this integral is odd as a function of x.

Therefore the difference is given by: (drawing a picture makes this easier)

$$\frac{2}{\pi}\int_0^\infty \frac{1-\cos(\pi w)}{w}\sin(xw)\,\mathrm{d}w = \begin{cases} 0 & x<-\pi \\ -1 & -\pi<x<0 \\ 1 & 0<x<\pi \\ 0 & x>\pi \end{cases},$$

and given by the midpoints of jumps at the jumps.

2.

(i) The point here is that the Fourier transform of the Gaussian is also a Gaussian, and the flatter the Gaussian is in the physical domain, the sharper is its Fourier transform:

$$\begin{aligned}\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} f(x)e^{-ix\xi}\,\mathrm{d}x =& \frac{1}{\sqrt{2\pi}}\int e^{-\lambda x^2-ix\xi-(i\xi/(2\sqrt{\lambda}))^2}e^{-\xi^2/(4\lambda)}\,\mathrm{d}x\\ =& \left(\frac{1}{\sqrt{2\pi}}\int e^{-(\lambda x+i\xi/(2\sqrt{\lambda}))^2}\,\mathrm{d}x\right)e^{-\xi^2/(4\lambda)}\\ =& \frac{C_{1/\sqrt{\lambda}}}{\sqrt{2\pi}}e^{-\xi^2/(4\lambda)}.\end{aligned}$$

(ii) This is of course also in the book:

$$\begin{aligned}\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} f(x)e^{-ix\xi}\,\mathrm{d}x =& \frac{1}{\sqrt{2\pi}}\int_0^{2\pi} e^{-ix\xi}\,\mathrm{d}x\\ =& \frac{1}{\sqrt{2\pi}i\xi}e^{-ix\xi}\Big|_0^{2\pi}\\ =& \frac{-2}{\sqrt{2\pi}\xi}e^{-i\pi\xi}\sin(\pi\xi).\end{aligned}$$

(iii) This is also a result of direct integration:

$$\begin{aligned}\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} f(x)e^{-ix\xi}\,\mathrm{d}x =& \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-\lambda|x|}e^{-ix\xi}\,\mathrm{d}x\\ =& \frac{1}{\sqrt{2\pi}}\int_{-\infty}^0 e^{\lambda x}e^{-ix\xi}\,\mathrm{d}x + \frac{1}{\sqrt{2\pi}}\int_0^\infty e^{-\lambda x}e^{-ix\xi}\,\mathrm{d}x\\ =& \frac{1}{\sqrt{2\pi}}\left(\frac{1}{\lambda-i\xi}-\frac{1}{-\lambda-i\xi}\right)\\ =& \frac{1}{\sqrt{2\pi}}\frac{2\lambda}{\lambda^2+\xi^2}.\end{aligned}$$

(iv) Finally, we integrate again:

$$\begin{aligned}\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} f(x)e^{-ix\xi}\,\mathrm{d}x =& \frac{1}{\sqrt{2\pi}}\int_{-1}^1 |x|e^{-ix\xi}\,\mathrm{d}x\\ =& -\frac{1}{\sqrt{2\pi}}\int_{-1}^0 xe^{-ix\xi}\,\mathrm{d}x + \frac{1}{\sqrt{2\pi}}\int_0^1 xe^{-ix\xi}\,\mathrm{d}x\\ =& \frac{1}{\sqrt{2\pi}i\xi}xe^{-ix\xi}\Big|_{-1}^0 + \frac{1}{\sqrt{2\pi}i\xi}\int_{-1}^0 e^{-ix\xi}\,\mathrm{d}x - \frac{1}{\sqrt{2\pi}i\xi}xe^{-ix\xi}\Big|_0^1 - \frac{1}{\sqrt{2\pi}(i\xi)}\int_0^1 e^{-ix\xi}\,\mathrm{d}x\\ =& \frac{1}{\sqrt{2\pi}i\xi}e^{i\xi} + \frac{1}{\sqrt{2\pi}(i\xi)^2}(1-e^{i\xi}) - \frac{1}{\sqrt{2\pi}i\xi}e^{-i\xi} - \frac{1}{\sqrt{2\pi}(i\xi)^2}(e^{-i\xi}-1)\\ =& \frac{2}{\sqrt{2\pi}\xi}\sin(\xi) - \frac{1}{\sqrt{2\pi}\xi^2}(2-2\cos(\xi))\\ =& \frac{1}{\sqrt{2\pi}}\left(\frac{2}{\xi}\sin(\xi)-\frac{4}{\xi^2}\sin^2(\xi/2)\right).\end{aligned}$$

3.

(i) simply put $-x$ in place of x in the integral in equation (4) of cap. 11 sec. 9, and observe that the integral separates to:

$$\begin{aligned} f(-x) =& \frac{1}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} f(v) e^{iw(-x-v)} \, \mathrm{d}v \, \mathrm{d}w \\ =& \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \left(\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(v) e^{-iwv} \, \mathrm{d}v \right) e^{-iwx} \, \mathrm{d}w \\ =& \hat{\hat{f}}(x). \end{aligned}$$

That is, the double transform is given by a reflection.

(ii) Using 2(iii), we see that

$$\frac{2\lambda}{\sqrt{2\pi}} f(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\lambda|x|} e^{-ix\xi} \, \mathrm{d}x.$$

Therefore from 3(i),

$$\begin{aligned} \hat{f}(y) =& \frac{\sqrt{2\pi}}{2\lambda} \cdot \frac{1}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-\lambda|x|} e^{-ix\xi} \, \mathrm{d}x \, e^{-i\xi y} \, \mathrm{d}\xi \\ =& \frac{\sqrt{2\pi}}{2\lambda} e^{-\lambda|-y|} \\ =& \frac{\sqrt{2\pi}}{2\lambda} e^{-\lambda|y|}. \end{aligned}$$

(iii) By the convolution properties of the Fourier transform,

$$\widehat{f * f} = \hat{f}^2 = \frac{\pi}{2\lambda^2} e^{-2\lambda|y|}.$$

The inverse transform of $e^{-2\lambda|y|}$ is its Fourier transform with a reflection.

$$\frac{1}{\sqrt{2\pi}} \frac{4\lambda}{4\lambda^2 + \xi^2}.$$

Multiplying in the constants we arrive at:

$$(f * f)(\xi) = \frac{\sqrt{2\pi}}{\lambda} \frac{1}{4\lambda^2 + \xi^2}.$$

4. Applying the manipulations already described in the question, this is a fairly straightforward problem:

$$\begin{aligned} & \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} F(s) e^{st} \, \mathrm{d}s \\ =& \frac{1}{2\pi} \int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(w) e^{-\gamma w} e^{-i\tau w} \, \mathrm{d}w \right) e^{\gamma t} e^{i\tau t} \, \mathrm{d}\tau \\ =& \frac{1}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} f(w) e^{\gamma(t-w)} e^{i\tau(t-w)} \, \mathrm{d}w \, \mathrm{d}\tau. \end{aligned}$$

Now apply the inversion formula for the function $g(w) = f(w) e^{\gamma(t-w)}$, and the result falls out.